Path-components of Morse mappings spaces of surfaces

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Abstract

Let M be a connected compact surface, P be either $\mathbb{R}^1$ or S^1, and $\mathcal{F}(M,P)$ be the space of Morse mappings $M \to P$ with compact-open topology. The classification of path-components of $\mathcal{F}(M,P)$ was independently obtained by S. V. Matveev and V. V. Sharko for the case $P = \mathbb{R}^1$, and by the author for orientable surfaces and $P = S^1$. In this paper we give a new independent and unified proof of this classification for all compact surfaces in the case $P = \mathbb{R}$, and for orientable surfaces in the case $P = S^1$. We also extend the initial author's proof to non-orientable surfaces.

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1. Introduction

Let M be a smooth (C^∞) connected compact surface with boundary ∂M (possibly empty) and P be a one-dimensional manifold, i.e. either the real line $\mathbb{R}^1$ or the circle S^1. Consider the subspace $\mathcal{F}(M,P)$ of $C^\infty(M,P)$ consisting of Morse mappings $M \to P$. It is well-known that $\mathcal{F}(M,P)$ is an everywhere dense open subset of $C^\infty(M,P)$ in the compact-open topology of $C^\infty(M,P)$. The homotopy type of this space is of great importance in differential topology and dynamical systems, see e.g. [H, I, HT, HH, KE, SV1, M, IS].

Recently, S. V. Matveev and V. V. Sharko [SV1] have obtained a full description of path-components of the space $\mathcal{F}(M,\mathbb{R}^1)$. Matveev's proof is included and generalized in the paper [KE] of E. Kudryavtseva to numerated Morse functions. Their proofs were independent and based on different ideas. The classification of path-components of $\mathcal{F}(M,S^1)$ for orientable surfaces was given in the author's Ph.D., see [M].

These results (which we will refer to as Main Theorem) can be summarized as follows: two Morse mappings $f,g : M \to P$ belong to same path-component of $\mathcal{F}(M,P)$ if and only if they are homotopic as continuous maps and have the same number of critical points at each index and the same sets of *positive* and *negative* boundary components (in the sense described below.)

In this paper we give a unified and independent proof of this theorem for all compact surfaces in the case $P = \mathbb{R}$. The case of Morse mappings

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$M \to S^1$ requires information on the subgroup of the mapping class group of M preserving a given element in the cohomology group $H^1(M, \mathbb{Z})$. We also find the generators of this group for orientable surfaces and extend the presented method to Morse mappings from orientable surfaces into S^1.

In fact, the proof given in [M] for this case almost literally extends to non-orientable surfaces as well. Since [M] was never published in English, we give this proof for all surfaces in Appendix. Thus the Main Theorem is proved here for all cases of M and P.

Our approach has a relation to the paper [HT] of A. Hatcher and W. Thurston, who used deformations of Morse functions to construct a representation for the mapping class group of a surface. In contrast to this approach, we exploit generators of this group to find a deformation between Morse mappings in $\mathcal{F}(M, P)$. The principal observation is that "elementary diffeomorphisms" like Dehn twists, boundary and crosscap slides generating mapping class groups of surfaces preserve certain Morse functions.

2. Preliminaries

Let M be a compact surface. A surface obtained by shrinking every connected component of M to a point will be denoted by $\widehat{M}$. Thus $\widehat{M}$ is closed and is homeomorphic with a connected sum of the form either $S^2 \mathop{\#}\limits_{i=1}^{g} T^2$ (orientable case, $g \geq 0$) or $\mathop{\#}\limits_{i=1}^{g} \mathbb{P}^2$ (non-orientable case, $g \geq 1$). In each of the cases the number g is called the germ of M. All homology and cohomology groups will be taken with integer coefficients. The term *simple closed curve* will be abbreviated to SCC. The circle S^1 will be regarded as the subset $\{z \in \mathbb{C} : |z| = 1\}$ of the complex plane $\mathbb{C}$. For a topological space X let $\#[X]$ denote the number of its connected components.

2.1. **Morse mappings.** Let us fix, once and for all, an orientation of P. Consider a smooth mapping $f : M \to P$. A point $z \in M$ is *critical* for f if $df(z) = 0$. A critical point z of f is *non-degenerate* if the Hessian of f at z is non-degenerate. Suppose that z is a non-degenerate critical point of f. Then by Morse lemma there are embeddings $p : \mathbb{R}^2 \to M$ and $q : \mathbb{R}^1 \to P$ onto open neighborhoods of z and $f(z)$ respectively such that $p(0,0) = z$, $q(0) = f(z)$, q preserves orientation, and $q^{-1} \circ f \circ p(x, y) = \pm x^2 \pm y^2$. The number of minuses in this representation does not depend on a particular choice of such embeddings and is called the *index* of a critical point z.

A C^∞-mapping $f : M \to P$ is *Morse* if the following conditions hold:

(1) all critical points of f are non-degenerate and belong to the interior of M;
(2) f is constant at each boundary component of M while its values on different components may differ each from other.

The subspace of $C^\infty(M, P)$ consisting of Morse mappings will be denoted by $\mathcal{F}(M, P)$. We endow $C^\infty(M, P)$ with the compact-open topology. Then this topology induces some topology on $\mathcal{F}(M, P)$.

2.2. **Σ-homotopies.** Let $f, g \in \mathcal{F}(M,P)$ be two Morse mappings and $\phi : [0,1] \to C^\infty(M,P)$ be a path between them in the space of Morse mappings, thus ϕ is continuous, $\phi(0) = f$, $\phi(1) = g$ and $\phi(t)$ is Morse for all $t \in [0,1]$. Then ϕ yields a continuous mapping (homotopy) $F : M \times I \to P$ such that $F_0 = f$, $F_1 = g$, and F_t is Morse for all $t \in I$. In particular, F is C^∞ in $x \in M$ but may be just continuous in $t \in [0,1]$. Conversely, every such mapping F gives rise to a path between f and g in $\mathcal{F}(M,P)$.

We will call the mapping F a Σ-*homotopy* or Σ-*deformation* between f and g and write $f \overset{F_t}{\sim} g$. The term $f \overset{\Sigma}{\sim} g$ will also be used to indicate that f and g are Σ-homotopic.

Remark 2.3. In [SV1, KE] Σ-homotopies are called *isotopies* of Morse functions. We will use another term in order to avoid confusions with isotopies of diffeomorphisms.

2.4. **Invariants of Σ-homotopies.** Let $f \in \mathcal{F}(M,P)$. The objects (i) homotopy class, (ii) number of critical points in each index, and (iii) positive and negative boundary components are invariant under Σ-homotopies of f.

2.4.1. *Homotopy class.* First suppose that $P = S^1$. Let $\xi \in H^1(S^1)$ be a generator defining the chosen orientation of S^1.

If $f : M \to S^1$ is a continuous mapping, then the correspondence $f \mapsto f^*(\xi) \in H^1(M)$ yields a bijection between the set of homotopy classes of mappings $[M, S^1]$ and the cohomology group $H^1(M)$. Since by our definition Morse mappings are constant at the connected components of M, it follows that the set of homotopy classes of Morse mappings $M \to S^1$ is bijective to the group $H^1(\widehat{M})$ for the corresponding closed surface $\widehat{M}$.

Let g be a genus of M. A simple calculation shows that $H^1(\widehat{M})$ is isomorphic with $\mathbb{Z}^r$, where r is either $2g$ or $g-1$ provided M is orientable or not. Let us fix a basis for $H^1(\widehat{M})$. Then the homotopy class of f is an integer vector

$$(q_1, \dots, q_r) = f(\xi) \in H^1(\widehat{M}) = \mathbb{Z}^r.$$

For $P = \mathbb{R}$ we will assume that $(q_1, \dots, q_r) = (0, \dots, 0)$.

2.4.2. *Number of critical points in each index.* Denote by $c_i(f) = c_i$, $(i = 0, 1, 2)$ the number of critical points of f of index i. Then by Morse equalities we have

$$c_0(f) + c_1(f) - c_2(f) = \chi(M). \tag{2.1}$$

2.4.3. *Positive and negative components of ∂M.* Let V be a component of ∂M, $z \in V$ and $\xi \in TM_z$ be a tangent vector at z directed *outward* M. Denote by $\varepsilon_f(V)$ the *sign* of the value $df(z)\xi$. Since f has no critical points on V, we see that $\varepsilon_f(V) = \pm 1$ and does not depend on a particular choice of a point $z \in V$ and a vector $\xi \in TM_z$ as above. Thus we get a function $\varepsilon_f : \pi_0 \partial M \to \{\pm 1\}$. We may also think of ε_f as an element of $\{\pm 1\}^b$, where b is the number of connected components of ∂M.

We will call V either *f-positive* or *f-negative* in accordance with $\varepsilon_f(V)$. Let $\partial_+ M$ (resp. $\partial_- M$) be the union of f-positive (resp. f-negative) boundary components of ∂M, and let b_+ (b_-) denote the numbers of these components.

The following collection of numbers

$$K(f) = \{q_1, \dots, q_r,\ c_0, c_1, c_2,\ \varepsilon_f\}$$

will be called the *critical type* of a Morse mapping f. It can be regarded as a point in $\mathbb{Z}^r \times \mathbb{N}_0^3 \times \{\pm 1\}^b$ belonging to the "hyperplane" defined by Eq. (2.1), where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If we choose another orientation of P, then $c_0(f)$ exchanges with $c_2(f)$, $c_1(f)$ remains unchanged, ε_f and every q_i change their signs.

Our aim is to give a new proof of the following theorem:

Main Theorem (Matveev [KE], Sharko [SV1], Maksymenko [M])**.** *Two Morse mappings $f, g : M \to P$ belong to the same path-component of $\mathcal{F}(M, P)$ if and only if $K(f) = K(g)$, i.e. they are homotopic, have the same numbers of critical points in each index, and the same sets of positive and negative components of ∂M.*

The necessity is obvious therefore we confine ourself by the sufficiency. Let us briefly review the known proofs of this theorem. First consider the case $P = \mathbb{R}^1$. Let f and g be two Morse functions with equal critical types. In the both proofs [KE, SV1] the problem was reduced to minimal Morse functions with no critical points of index 0 and 2.

Let F be a gradient-like vector field for a minimal Morse function f. Consider a union of f-negative boundary components of M with trajectories of F that finish at critical points of f. This set is called a *spine* of M. Matveev (see Kudryavtseva [KE]) notes that the space of Morse functions with isotopic spines is path-connected. He further suggested elementary transformations of spines which induce Σ-homotopies of Morse function and showed that any two spines can be connected by a finite sequence of these transformations.

Sharko [SV1] reduced the problem to minimal Morse functions on a surface M with only one positive and only one negative boundary component. Such a surface can be regarded as a "framed" chords diagram in which the union of all chords and a negative boundary component constitute the spine of M. Notice that $\pi_1 M$ is free. Choose a basis of this group. Then the edges of any other chords diagram in M can be written down as words in the terms of a given basis. These words also form the basis of $\pi_1 M$ and determine chords diagram up to equivalence. Moreover, by the well-known Nielsen theorem any two bases of a finitely generated free group are related by a finite sequence of Nielsen transformations. Sharko proved that Nielsen transformations yield Σ-homotopies between corresponding Morse functions, and that Morse functions with equivalent diagrams are Σ-homotopic.

The extension of the proof of [M] for $P = S^1$ and all surfaces is given in the Appendix.

2.5. **Plan of the present proof.** First the problem will be reduced to the case when $g = f \circ h$, where h is a diffeomorphism of M and f is of a special "canonical" form. It is convenient to say that a diffeomorphism h is f-admissible if $f \overset{\Sigma}{\sim} f \circ h$. Using a special type of f, we will choose system of generators for $\mathcal{M}(M)$ and show that if $P = \mathbb{R}$, then all of them are f-admissible. This will prove the Main Theorem for this case.

For the case $P = S^1$, M is orientable, and f is not null-homotopic we shall see that one of the generators chosen above is not f-admissible. Nevertheless, since f and $f \circ h$ are homotopic, it will be possible to reduce the problem to the case when h acts trivially on the homology group $H_1(M, \partial M)$, i.e. h belongs to the Torelli group of M. Generators of this group are known from [P], [J], [MG]. This information will allow us to show that $f \overset{\Sigma}{\sim} f \circ h$.

2.6. **Structure of the paper.** In Section 3 we prove some technical results concerning to Morse mappings to the circle. In Section 4 we recall the definition of the Kronrod-Reeb graph of a Morse mapping and define "canonical" Morse mappings. In Section 5 we reduce the Main Theorem to the case when f is canonical and g differs from f by a diffeomorphism. This was done by Kudryavtseva in [KE] for Morse functions. We consider the case $P = S^1$. In Section 6 we show that elementary diffeomorphisms generating mapping class group $\mathcal{M}(M)$ of M (Dehn twists, boundary and crosscap slides) preserve certain Morse functions. In Section 7 we recall the generators of mapping class groups for surfaces with boundary. Every canonical Morse mapping gives a "canonical" set of such generators whose admissibility (or nonadmissibility) for this map is almost obvious. We also complete the Main Theorem for $P = \mathbb{R}$ (statement (i) of Lemma 7.6).

In Section 8 we give the plan of the proof of Main Theorem for the case M is orientable and $P = S^1$. For this in Section 9 we consider the stabilizers of elements of $\mathbb{Z}^{2g}$ with respect to the action of the symplectic groups $Sp_{2g}(\mathbb{Z})$, in Section 10 we study minimal Morse functions. Section 11 includes one technical lemma. Finally in Sections 12-14 we complete the proof.

3. Cutting M along a regular level-set of f.

We prove here two lemmas which will be used in the proof of Proposition 5.4.

Let c be a regular value of a Morse mapping $f : M \to S^1$. Then $f^{-1}(c)$ is a disjoint union of SCCs on M. Suppose that $f^{-1}(c) \cap \partial M = \varnothing$. We cut M along $f^{-1}(c)$ and denote the new surface by $\widetilde{M} = \widetilde{M}(f, c)$. Similarly, we cut S^1 at $f(c)$ and obtain $[0, 1]$. Let $p : \widetilde{M} \to M$ and $q : [0, 1] \to S^1$ be the corresponding factor-maps, where $q(t) = e^{2\pi i t}$, $t \in [0, 1]$. Then there exists a Morse function $\widetilde{f} : \widetilde{M} \to [0, 1]$ such that the following diagram is

commutative:

$$
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\widetilde{f}} & [0,1] \\
{\scriptstyle p}\Big\downarrow & & \Big\downarrow{\scriptstyle q} \\
M & \xrightarrow{f} & S^1.
\end{array}
\tag{3.1}
$$

Thus

$$
f(x) = \exp\left(2\pi i \widetilde{f}(p^{-1}(x))\right), \ \forall x \in M. \tag{3.2}
$$

Denote $B_0 = \widetilde{f}^{-1}(0)$, $B_1 = \widetilde{f}^{-1}(1)$, and $B = B_0 \cup B_1$. Then there is a natural corresponding between Σ-homotopies $\widetilde{f}_t$ of $\widetilde{f}$ with respect to some neighborhood of B and Σ-homotopies f_t of f with respect to some neighborhood of γ. The corresponding maps $\widetilde{f}_t$ and f_t are related by the commutative diagram (3.1).

Since M is connected, it follows that every connected component X of $\widetilde{M}$ intersects B. However, it is possible that $X \cap B_i = \varnothing$ for some $i = 0, 1$. Thus the components of $\widetilde{M}$ can be divided into the following mutually disjoint sets

$$
Q_0 = Q_0(f,c), \qquad Q_0^1 = Q_0^1(f,c), \qquad Q^1 = Q^1(f,c) \tag{3.3}
$$

that (respectively) intersect only B_0, intersect both sets B_0 and B_1, and intersect only B_1.

It follows that for every connected component X of $Q_0^1(f,c)$ and $t \in [0,1]$ we have $X \cap f^{-1}(t) \neq \varnothing$.

Lemma 3.1. (1) *Let V be an $\widetilde{f}$-positive (resp. $\widetilde{f}$-negative) component of $\partial\widetilde{M}$ and $v = \widetilde{f}(V)$. Then for every $w > v$ (resp. $w < v$) there exists a Σ-homotopy $\widetilde{f}_t$ changing $\widetilde{f}$ only in arbitrary small neighborhood of V and such that $\widetilde{f}_1(V) = w$, see Figure 3.1a).*

(2) *Let X be a connected component of $\widetilde{M}$. For every $w \in (0,1)$ there exists a Σ-homotopy $\widetilde{f}_t : \widetilde{M} \to [0,1]$ such that $\widetilde{f}_0 = \widetilde{f}$, $\widetilde{f}_t = \widetilde{f}$ on $(\widetilde{M} \setminus X) \cup B$, and $\widetilde{f}_1^{-1}(\frac{1}{2}) \cap X = \widetilde{f}^{-1}(w) \cap X$, see Figure 3.1b).*

(3) *Let X be a connected component of $\widetilde{M}$. Then there exists a Σ-homotopy $\widetilde{f}_t : \widetilde{M} \to [0,1]$ with $\widetilde{f}_0 = \widetilde{f}$ and $\widetilde{f}_t = \widetilde{f}$ on $(\widetilde{M} \setminus X) \cup B$, such that $\widetilde{f}_1^{-1}(\frac{1}{2}) \cap X = \varnothing$, whenever $X \subset Q_0 \cup Q^1$, and $\#[\widetilde{f}_1^{-1}(\frac{1}{2}) \cap X] = 1$, whenever $X \subset Q_0^1$.*

Proof. (1) Suppose that V is an $\widetilde{f}$-positive component of $\partial\widetilde{M}$. By definition, $\widetilde{f}$ has no critical points on V. Then there exist an $\varepsilon > 0$, a neighborhood N of V, and a diffeomorphism $h : S^1 \times (v - 2\varepsilon, v] \to N$ such that $h(S^1 \times \{v\}) = V$ and $\widetilde{f} \circ h(x,t) = t$ for $(x,t) \in S^1 \times (v - 2\varepsilon, v]$.

Let H_t be an isotopy of $\mathbb{R}$ fixed on $(-\infty, v - \varepsilon]$ and such that $H_1(v) = w$. Then the Σ-homotopy $\widetilde{f}_t$ defined by the formulas: $\widetilde{f}_t(x) = \widetilde{f}(x)$ for $x \in M \setminus N$

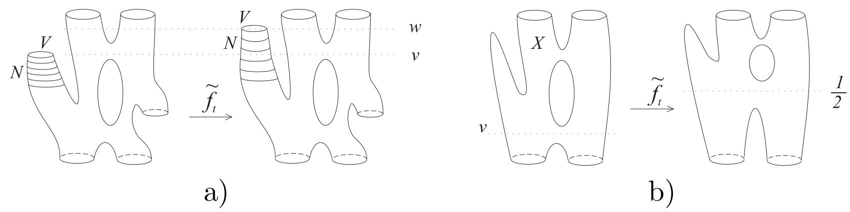


FIGURE 3.1.

and $\widetilde{f}_t(x) = H_t \circ \widetilde{f}(x)$ for $x \in N$ satisfies the statement (1) of lemma. The proof for $\widetilde{f}$-negative components is similar.

(2) Notice, that for any $v \in (0,1)$ there exists an isotopy H_t of $\mathbb{R}^1$ fixed near 0 and 1 and such that $H_1(s) = \frac{1}{2}$. Then the Σ-homotopy $\widetilde{f}_t : \widetilde{M} \to [0,1]$ defined by the formulas: $\widetilde{f}_t = H_t \circ \widetilde{f}$ on X and $\widetilde{f}_t = \widetilde{f}$ on $\widetilde{M} \setminus X$ satisfies the statement (2) of lemma.

(3) It follows from the definition that for every connected component X of $Q_0 \cup Q^1$ there exists a number $v \in (0,1)$ such that $\widetilde{f}^{-1}(v) \cap X = \varnothing$. Therefore, if $X \subset Q_0 \cup Q^1$, then our statement follows from (2).

Let $X \subset Q_0^1$. If for some $i = 0,1$ the intersection $X \cap B_i$ is connected, then for every t in some neighborhood of i we have that $X \cap \widetilde{f}^{-1}(t)$ is connected. By (1) of this lemma we can choose $t = \frac{1}{2}$.

Suppose now that the intersections $X \cap B_i$, $i = 0,1$ are not connected. By (1) and (2) we assume that

$$0 < \widetilde{f}(p^{-1}(\partial_- M) \cap X) < \frac{1}{4} < \widetilde{f}(\Sigma_{\widetilde{f}} \cap X) < \frac{1}{2} < \widetilde{f}(p^{-1}(\partial_+ M) \cap X) < 1,$$

where $\Sigma_{\widetilde{f}}$ is the set of critical points of $\widetilde{f}$. Thus all critical values of $\widetilde{f}|_X$ belong to $(\frac{1}{4}, \frac{1}{2})$; the values on $\widetilde{f}$-negative boundary components of X except for $\widetilde{f}(X \cap B_0) = 0$ are in $(0, \frac{1}{4})$; and the values on $\widetilde{f}$-positive boundary components of X except for $\widetilde{f}(X \cap B_1) = 1$ are in $(\frac{1}{2}, 1)$. In particular, $\frac{1}{2}$ is a regular value of $\widetilde{f}$.

Denote $n = \#[\widetilde{f}^{-1}(\frac{1}{2})]$ and suppose that $n > 1$. Our object is to reduce n. Let F be a gradient-like Morse-Smale vector field of X for the function $\widetilde{f}|_X$. It follows from Morse theory that the union of $\widetilde{f}|_X$-positive boundary components $\partial_+ X$ with the set of trajectories that start at saddle critical points of $\widetilde{f}|_X$ and finish at $\partial_+ X$ is a strong deformation retract of X. Since X is connected, we see that there exists a saddle critical point z of $\widetilde{f}|_X$ such that the trajectories starting from z finish at different components of $\partial_+ X$. We denote these trajectories by ω_1 and ω_2.

Then (Milnor [MJ1], Theorem 4.1) there exists a Σ-homotopy $\widetilde{f}_t$ of $\widetilde{f}_0 = \widetilde{f}|_X$ that changes $\widetilde{f}|_X$ only in arbitrary small neighborhood of $(\omega_1 \cup \omega_2) \cap \widetilde{f}^{-1}(\frac{1}{4}, \frac{1}{2}]$ such that $\frac{1}{2} < \widetilde{f}_1(z) < 1$, but $\widetilde{f}_1(z') < \frac{1}{2}$ for all other critical point z' of $\widetilde{f}_1$. It follows that $\frac{1}{2}$ is a regular value for $\widetilde{f}_1$ and the level-set $\widetilde{f}_1^{-1}(\frac{1}{2})$

has precisely $n-1$ connected components. Now (3) follows by induction on n. $\square$

Lemma 3.2. *Every Morse mapping $f : M \to S^1$ is Σ-homotopic to a Morse mapping g such that for some regular value c of g we have:*
(A) *if f is null-homotopic, then $g^{-1}(c) = \varnothing$;*
(B) *otherwise, $\#[g^{-1}(c)]$ is equal to the index of $f_*(H_1(M))$ in $H_1(S^1)$.*

Proof. Let c be a regular value of f such that $f^{-1}(c) \cap \partial M = \varnothing$ and let $n = \#[f^{-1}(c)]$. We cut M and obtain the surface $\widetilde{M} = \widetilde{M}(f, c)$ and the function $\widetilde{f} : \widetilde{M} \to [0, 1]$ as above.

By Lemma 3.1, if $Q_0 \cup Q^1 \neq \varnothing$ or if for some connected component X of Q_0^1 the intersection $X \cap B_0$ has more than one component, then there exists a Σ-homotopy $\widetilde{f}_t$ of $\widetilde{f}$ with respect to some neighborhood of B such that $\#[\widetilde{f}_1^{-1}(\frac{1}{2})] < n$. As noted above, this Σ-homotopy yields a Σ-homotopy f_t of $f = f_0$ to a Morse mapping f_1 with respect to some neighborhood of $f^{-1}(c)$ such that $\#[f_1^{-1}(c_1)] < n$, where $c_1 = q(\frac{1}{2})$ is a regular value of f_1.

Repeating these arguments for f_1 and c_1, and using induction in n we will obtain a Morse mapping f_k and its regular value c_k such that either (i) $f_k^{-1}(c_k) = \varnothing$ or (ii) $Q_0(f_k, c_k) = Q^1(f_k, c_k) = \varnothing$ and for every connected component X of $Q_0^1 = \widetilde{M}(f_k, c_k)$ the intersection $X \cap B_i(f_k, c_k)$ is non-empty and connected, whence it is an SCC.

Suppose that f_k is null-homotopic. Then f_k lifts to a Morse function $\widetilde{f}_k : M \to \mathbb{R}^1$ which must have global minimum and maximum. Therefore, if $f_k^{-1}(c_k) \neq \varnothing$ (case (ii)), then $Q_0(f_k, c_k) \cup Q^1(f_k, c_k) \neq \varnothing$, which contradict to (ii). Hence, $f_k^{-1}(c_k) = \varnothing$. This proves (A).

Suppose f_k is not null-homotopic. For the convenience we denote f_k by f and c_k by c. We will now lift f onto the covering of S^1 corresponding to the subgroup $f(H_1(M))$ of $H_1(S^1)$. Let $m = \#[\widetilde{M}]$ and $p_m : S^1 \to S^1$ be the m-sheet-covering of S^1 defined by the formula $p_m(e^{2\pi i t}) = e^{m 2\pi i t}$, $t \in [0, 1]$.

First notice, that the set of connected components of $\widetilde{M}$ admits a natural *cyclic* ordering. Indeed, let X_0 be any component of $\widetilde{M}$. If X_k, $(k \geq 0)$ is defined, then there exists a unique connected component X_{k+1} of $\widetilde{M}$ such that $p(X_{k+1} \cap B_0) = p(X_k \cap B_1)$. Since M is connected, it follows that *every* connected component of $\widetilde{M}$ is numbered in this way.

Then the following formula defines a lifting $\bar{f} : M \to S^1$ of f onto the m-sheet covering of S^1:

$$\bar{f}(x) = \exp \frac{2\pi i}{m} \left(\widetilde{f}(p^{-1}(x)) + k \right), \quad x \in p(X_k),\ k = 0, \dots, m-1,$$

i.e. $p_m \circ \bar{f} = f$.

Finally, let us prove that the homomorphism $\bar{f}_* : H_1(M) \to H_1(S^1)$ is onto. This will imply that index of $f(H_1(M))$ in $H_1(S^1)$ is m. For every $k = 0, \dots, m-1$ let $\omega_k : [0, 1] \to X_k$ be a simple arc which is transversal to level-sets of $\widetilde{f}$ and such that $\widetilde{f}(\omega_k(t)) = t$, $p(\omega_k(1)) = p(\omega_{k+1}(0))$ and

$p(\omega_{m-1}(1)) = p(\omega_0(0))$. Evidently, these arcs constitute an SCC ω on M such that the restriction $\bar{f}|_\omega$ is homeomorphism, whence $\bar{f}_*$ is onto. Thus (B) is proved. □

3.3. **Orientation of level-sets of f.** Suppose that M is orientable. Let $c \in S^1$ be a regular value of a Morse mapping $f : M \to S^1$, $L = f^{-1}(c)$ be the corresponding level-set of f, and F be a gradient vector field for f taken in some Riemannian metric on M. Then the orientation of M together with F yields an orientation of L so that the homology class of an oriented cycle $[f^{-1}(c)] \in H_1(M, \partial M)$ does not depend on a particular choice of a regular value c and even on the homotopy class of f. For every $x \in L$ let v_x be a tangent vector to L at x such that the pair $(v_x, \mathrm{grad} f(x))$ gives a positive orientation of M. Then the orientation of L defined by v_x satisfies the conditions of the previous sentence.

Let $\xi \in H^1(S^1)$ be a generator that defines the positive orientation of S^1 and ω be an intersection form on $H_1(M, \partial M)$. Then for every oriented SCC $\gamma : S^1 \to M$ regarded as an element of $H_1(M)$ we have

$$f(\xi)(\gamma) = \langle L, \gamma \rangle = \deg(f|_\gamma). \tag{3.1}$$

Since f is constant on boundary components of M and is not null-homotopic it follows that $f(\xi) \neq 0$ in $H^1(M, \partial M)$. The intersection form ω on M yields an isomorphism $\phi : H^1(M, \partial M) \to H_1(M, \partial M)$ which by Eq. (3.1), maps $f(\xi)$ to the homology class $[L]$.

In particular, if $h : M \to M$ is a diffeomorphism such that $f \circ h$ and f are homotopic, then it follows that $h^*(f(\xi)) = f(\xi)$ in $H^1(M, \partial M)$ and $h_*([L]) = [L]$ in $H_1(M, \partial M)$.

4. Kronrod-Reeb graph of a Morse mapping

Let $f : M \to P$ be a Morse mapping, $c \in P$, and γ be a connected component of $f^{-1}(c)$. We call γ *regular* if it contains no critical points of f; otherwise γ is *critical*.

Consider the partition of M by the connected components of level-sets of f. The factor-space Γ_f of M by this partition has the structure of a one-dimensional CW-complex and is called the *Kronrod-Reeb graph* or *KR*-graph of f (see e.g. [KA, KE, SV2]). There is a unique decomposition

$$f : M \xrightarrow{f^*} \Gamma_f \xrightarrow{f_\Gamma} P,$$

where f^* is a factor map and for every *open* edge e of Γ_f the restriction $f_\Gamma|_e$ is a local homeomorphism. Notice that the orientation of P yields a unique orientation of e preserved by f_Γ. The mapping f_Γ will be called *KR-map* associated with f.

The vertices of Γ_f correspond to the critical components of level-sets of f and to the boundary circles of M. The last type vertices will be denoted

on the *KR*-graph by circles $\circ$ (see e.g. Figure 4.1). Notice that for non-orientable surfaces *KR*-graphs can possess vertices of degree 2 (e.g. [KE]). We will denote these vertices by stars $*$.

Let $f, g : M \to P$ be Morse mappings. By *isomorphism* between their *KR*-graphs we will mean a homeomorphism $\Gamma_g \to \Gamma_f$ preserving orientations of edges and the sets of $\circ$- and $*$-vertices.

We will say that their *KR*-maps f_Γ and g_Γ are *equivalent* provided there exist a preserving orientation diffeomorphism ϕ of P and an isomorphism $\alpha : \Gamma_g \to \Gamma_f$ such that in the following diagram the right square is commutative:

$$\begin{array}{ccccc} M & \xrightarrow{g^*} & \Gamma_g & \xrightarrow{g_\Gamma} & P \\ {\scriptstyle h}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \phi}\downarrow \\ M & \xrightarrow{f^*} & \Gamma_f & \xrightarrow{f_\Gamma} & P \end{array} \tag{4.2}$$

The mappings f and g are said to be *equivalent* provided there exists a diffeomorphism h of M such that $f \circ h = \phi \circ g$. In this case there is a unique equivalence α between *KR*-maps of f and g such that the whole diagram (4.2) is commutative.

A Morse mapping f is called *generic* if every level-set of f contains at most one critical point. Let f be a generic Morse mapping. If M is orientable, then the degree of each vertex of Γ_f is either 1 or 3. If M is non-orientable, then Γ_f can possess vertices of degree 2.

The following lemma is well-known. Its different variants can be found in [BF, KE, K, SV2].

Lemma 4.1. *Two generic Morse mappings f and g having equivalent KR-maps are equivalent.* $\square$

We say that a Morse mapping f is *canonical* if its *KR*-map is equivalent to that drawn in Figures 4.1 or 4.2.

First consider the case $P = \mathbb{R}$, see Figure 4.1. The part of *KR*-graph under the rectangle corresponds to the following cases of M:

a) M is orientable;

b) M is non-orientable of odd genus g;

c) M is non-orientable of even genus g;

d) M is non-orientable, $g \geq 3$ and is odd. In this case we will use two types of canonical Morse functions shown in Figure 4.1. They are related by a Σ-homotopy, see [KE].

For the case $P = S^1$ a canonical Morse mapping $f : M \to S^1$ can be described as follows: there is a regular value c of f such that $\gamma = f^{-1}(c)$ is a SCC. Moreover, if we cut M along γ, then the restriction of $f : M \backslash \gamma \to S^1 \backslash c$ is a canonical Morse function. Its *KR*-graph is hidden behind the rectangle, see Figure 4.2.

Notice also, that a canonical Morse mapping is generic and the homomorphism $f_* : H_1(M) \to H_1(S^1)$ is onto.

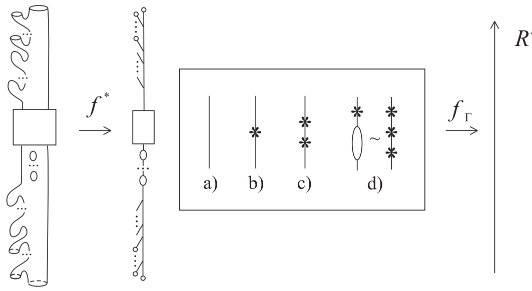


FIGURE 4.1. *KR*-graphs and *KR*-maps of a canonical Morse function $M \to \mathbb{R}$.

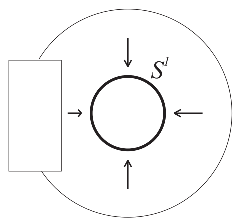


FIGURE 4.2. *KR*-graphs and *KR*-maps of a canonical Morse mapping $M \to S^1$.

Lemma 4.2. *Let $f, g : M \to P$ be two canonical Morse mappings of same critical type $K(f) = K(g)$. Then they are equivalent.*

Moreover, there is a Σ-homotopy of g to a canonical Morse mapping g_1 such that $g_1 = f \circ h$, where h is a diffeomorphism of M.

Proof. Evidently, *KR*-graph and *KR*-map of a canonical Morse mapping is determined by the numbers c_0, c_2, b_+, b_- and the (orientable or non-orientable) genus g of M. Notice that c_1 is related to these numbers via Euler characteristic.

Hence the condition $K(f) = K(g)$ implies that *KR*-maps of f and g are equivalent. Then by Lemma 4.1, f and g are equivalent, i.e. $p \circ g = f \circ h$, where p is a preserving orientation diffeomorphism of P and h is a diffeomorphism of M. It follows that p is isotopic to id_M. Let p_t be an isotopy of $p = p_1$ to $\mathrm{id}_M = p_0$. Then $g_t = p_t \circ g$ is a Σ-homotopy of $g = g_0$ to $g_1 = p_1 \circ g = p \circ g = f \circ h$. □

5. REDUCTION OF THE PROBLEM

Let $f, g : M \to P$ be two Morse mappings such that $K(f) = K(g)$. We have to prove that $f \overset{\Sigma}{\sim} g$.

In this section we reduce the proof of Main Theorem to the case when f and g are canonical, and $g = f \circ h$, where h is a diffeomorphism of M. This was done in [KE] for the case $P = \mathbb{R}$. Let $P = S^1$.

5.1. **Step 1.** It may be assumed that the homomorphism $f_* = g_* : H_1(M) \to H_1(S^1)$ is surjective. In particular, f and g are not null-homotopic. This also implies that M is neither a sphere nor a projective plane (with holes if $\partial M \neq \varnothing$).

Indeed, suppose that the homomorphism $f_* = g_*$ is not onto. Let $p : \widetilde{S} \to S^1$ be the covering of S^1 corresponding to the subgroup $f_*(H_1(M)) \subset H_1(S^1) = \pi_1(S^1)$ and $\widetilde{f}, \widetilde{g} : M \to \widetilde{S}$ be some liftings of f and g respectively which are evidently Morse.

Lemma 5.2. *$f \overset{\Sigma}{\sim} g$ iff $\widetilde{f} \overset{\Sigma}{\sim} \widetilde{g}$.* □

The proof is easy and is left to the reader. It can be found in [M].

5.3. **Step 2.** We may assume that f and g are canonical due to the following statement:

Proposition 5.4 ([KE])**.** *Every Morse mapping $f : M \to P$ such that the homomorphism $f_*(H_1(M)) \subset H_1(S^1) = \pi_1(S^1)$ is onto is Σ-homotopic to a canonical one.*

It follows from this proposition that $f \overset{\Sigma}{\sim} f_1$ and $g \overset{\Sigma}{\sim} g_1$, where f_1 and g_1 are canonical Morse mappings of same critical type $K(f) = K(g)$. Then by Lemma 4.2 $g_1 = f_1 \circ h$, where h is a diffeomorphism of M.

Proof. As noted above, this statement is proved in [KE] (Lemma 10) for closed surfaces and $P = \mathbb{R}$. The proof easily extends to surfaces with boundary. Suppose that $P = S^1$. Since f_* is onto, it follows from Lemma 3.2, that f is Σ-homotopic to a Morse mapping f_1 such that $\alpha = f_1^{-1}(c)$ is an SCC, where c is a regular value of f_1. Cutting M along α as in Section 3 we obtain a surface $\widetilde{M}$ and a function $\widetilde{f} : \widetilde{M} \to [0, 1]$. Then by the $\mathbb{R}$-case of this proposition $\widetilde{f}$ is Σ-homotopic with respect to a neighborhood of B to a canonical Morse function. This Σ-homotopy yields a Σ-homotopy of f to a canonical Morse mapping. □

6. Admissible diffeomorphisms and curves

Definition 6.1. Let $f : M \to P$ be a Morse mapping. A diffeomorphism $h : M \to M$ will be called *f-admissible* provided $f \circ h$ is Σ-homotopic to f. Notice that f-admissibility implies that h preserves the sets of f-positive and f-negative components of ∂M and that f and $f \circ h$ are homotopic.

Let $\mathcal{A}(f) \subset \mathcal{D}M$ be the set of all f-admissible diffeomorphisms, $\mathcal{D}_{\mathrm{id}}M$ be the identity component of $\mathcal{D}M$, and $C(f)$ be the path-component of f in $\mathcal{F}(M, P)$.

Lemma 6.2. *$\mathcal{A}(f)$ is a group consisting of full isotopy classes, i.e. $\mathcal{D}_{\mathrm{id}}M \subset \mathcal{A}(f)$. Moreover, if $g \in C(f)$, then $\mathcal{A}(g) = \mathcal{A}(f)$.*

Proof. Suppose that $p, q \in \mathcal{A}(f)$ and let $f \overset{\Phi_t}{\sim} f \circ p$ and $f \overset{\Psi_t}{\sim} f \circ q$ be Σ-homotopies. Then $p \circ q$ and $p^{-1} \in \mathcal{A}(f)$. Indeed,

$$f \overset{\Psi_t}{\sim} f \circ q \overset{\Phi_t \circ q}{\sim} f \circ p \circ q \quad \text{and} \quad f = f \circ p \circ p^{-1} \overset{\Phi_{1-t} \circ p^{-1}}{\sim} f \circ p^{-1}.$$

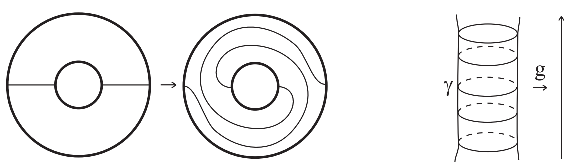
Thus $\mathcal{A}(f)$ is a group.

If $p \overset{H_t}{\sim} p_1$ is an isotopy, then the homotopy $f \overset{\Phi_t \circ H_t}{\sim} f \circ p_1$ is a Σ-homotopy. Thus $\mathcal{A}(f)$ consists of full isotopy classes.

Finally, if $f \overset{\Psi_t}{\sim} g$ is a Σ-homotopy, then $g \overset{\Psi_t}{\sim} f \overset{\Phi_t}{\sim} f \circ p \overset{\Psi_{1-t} \circ p}{\sim} g \circ p$. Hence $p \in \mathcal{A}(g)$, i.e. $\mathcal{A}(f) \subset \mathcal{A}(g)$. Similarly $\mathcal{A}(g) \subset \mathcal{A}(f)$. $\square$

We will now consider three types of "elementary diffeomorphisms" and show that they preserve certain simple Morse functions.

6.3. **Dehn twists.** Let γ be a two-sided oriented SCC in M. For definition of a Dehn twist along γ see e.g. [D, L1]. This diffeomorphism is supported in some neighborhood of γ and its effect on such a neighborhood is shown in Figure 6.1a).



a) b)

FIGURE 6.1. Dehn twist

Definition 6.4. Let γ be a two-sided SCC in $M \setminus \partial M$. We say that γ is *f-admissible* if f is Σ-homotopic to a Morse mapping g such that γ is a connected component of a regular level-set of g.

Lemma 6.5. *Let $\gamma \subset \mathrm{Int}M$ be a f-admissible oriented SCC in M. Then a Dehn twist t_γ along γ is f-admissible.*

Proof. Let $f \overset{F}{\sim} g$ be a Σ-homotopy such that γ is a connected component of a regular level-set of g. We will construct a Dehn twist t_γ along γ such that $g = g \circ t_\gamma$. Then t_γ is g-admissible, whence by (1) of Lemma 6.2 t_γ is also f-admissible.

Since γ is a regular component of a level set of g, then there is a regular neighborhood of γ which is diffeomorphic to $S^1 \times I$ and such that the function g is the projection to I, see Figure 6.1b). Then there is a Dehn twist t_γ along γ that preserves the sets of the form $S^1 \times \{t\}$. They are level-sets of g, whence t_γ preserves g. $\square$

6.6. **Boundary slides.** Let A be an annulus and C_0, C_1 be the connected components of ∂A. Divide C_1 into four arcs of equal length $l_1, \ldots, l_4$ so that l_1 is opposite to l_3 and l_2 to l_4. Let us identify the opposite points of l_1 and l_3. Then the quotient is a Möbius strip B with the hole $C_1' = l_2 \cup l_4$.

Let $\tau : A \to A$ be a half-Dehn twist along C_1, which exchanges l_1 with l_3 and l_2 with l_4 and is identity near C_0. Then τ yields a certain diffeomorphism ν of B that "rotates C_1' by π and fixes C_0", see Figure 6.1a).

Suppose that B is embedded to M so that C_1 is mapped onto a connected component C of ∂M. Then ν extends by the identity on all of M. This diffeomorphism is called a *boundary slide* of C along B.

Notice that our description of boundary slide differs from ones given in [KM, SB]. The advantage is an evidence of the symmetry of ν.

Now it is easy to see that there is a Morse function $f : B' \to [0, 1]$ having a unique critical point of index 1 and such that $f^{-1}(0) = C_0$, $f^{-1}(1) = C_1'$. Its critical level sets and the KR-graph are shown in Figure 6.1b).

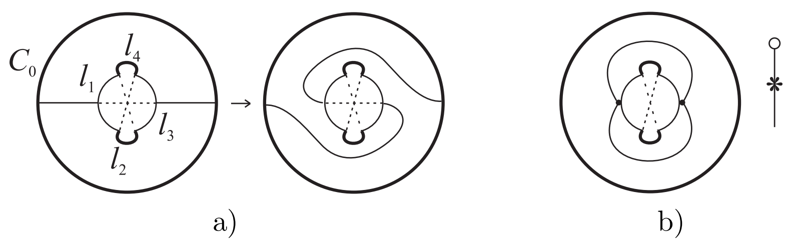


FIGURE 6.1. Boundary slide

The following lemma is obvious.

Lemma 6.7. *$f : M \to P$ be a Morse mapping on a non-orientable surface M. Suppose that KR-graph of f has an edge e such that one of its vertices v_1 has degree 2 and another one v_2 corresponds to the boundary component of M, see Figure 6.1b). Let N be a neighborhood of e containing no vertices of Γ_f but ∂e. Then $B = f_\Gamma^{-1}(N) \subset M$ is a Möbius band with hole and there exists a boundary slide $\nu : M \to M$ of $f_\Gamma^{-1}(v_2)$ along B such that $f \circ y = f$.* □

6.8. **Crosscap slides.** This type of diffeomorphisms was introduced by W. B. R. Lickorish [L2] and called an Y-diffeomorphism. In [KM, SB] the term *crosscap slide* is used. We recall the definition of this diffeomorphism (given in [BC]) via oriented double coverings.

Let K be a Klein bottle with two holes and $p : T \to K$ be its oriented double covering, where T is a torus with 4 holes. We can assume that T is embedded in $\mathbb{R}^3$ so that it is symmetrical with respect to the origin 0. In other words it is invariant under the involution $\xi(x, y, z) = (-x, -y, -z)$ of $\mathbb{R}^3$, see Figure 6.1a).

Let $V_1, \ldots, V_4$ be the connected components of ∂T numbered so that $\xi(V_1) = V_2$ and $\xi(V_3) = V_4$. Then there is a diffeomorphism $\widetilde{y} : T \to T$

which is fixed near $V_3 \cup V_4$, coincides with ξ near $V_1 \cup V_2$ and such that $\widetilde{y} \circ \xi = \xi \circ \widetilde{y}$. Thus y can be described as a "rotation" of T with respect to z-axis by π with fixed boundary components V_3 and V_4. For example, in Figure 6.1a) an arc and its image under ξ are shown. It follows that $\widetilde{y}$ induces some diffeomorphism y of K fixed near ∂K.

Suppose that $K \subset M$ is embedded in M. Then y extends by the identity to a diffeomorphism of M. Such a diffeomorphism of M is called *Y-diffeomorphism* or *crosscap slide* based in K.

Notice that there is a Morse function $\widetilde{f} : T \to \mathbb{R}$ with 4 critical points such that $\widetilde{f} \circ \widetilde{y} = \widetilde{f}$, see Figure 6.1a), where the critical level-sets of $\widetilde{f}$ are shown. Then $\widetilde{f}$ yields a unique Morse function $f : K \to \mathbb{R}$ having 2 critical points and such that $f \circ y = f$. The *KR*-graphs $\Gamma_{\widetilde{f}}$ and Γ_f of $\widetilde{f}$ and f are shown in Figure 6.1b).

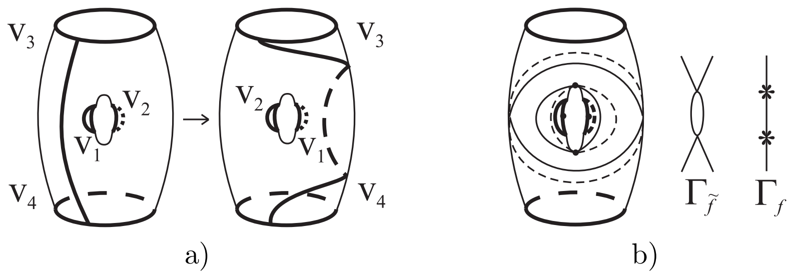


FIGURE 6.1. Crosscap slide on the orientable covering

Lemma 6.9. *Let $f : M \to P$ be a Morse mapping on a non-orientable surface M. Suppose that KR-graph of f has an edge e with vertices of degree 2. Let N be a neighborhood of e containing no vertices of Γ_f but ∂e. Then $K = f_\Gamma^{-1}(N) \subset M$ is a Klein bottle with two holes and there exists an Y-diffeomorphism $y : M \to M$ based in K such that $f \circ y = f$.* $\square$

7. Mapping class group of a surface with boundary

Let $\widehat{M}$ be a closed connected surface and $X = \{x_1, \dots, x_n\}$ be a set of mutually distinct points of $\widehat{M}$. The *extended mapping class group* $\mathcal{M}_n(M)$ of M is defined to be the group of isotopy classes of diffeomorphisms of $\widehat{M}$ which take X to itself. The *pure extended mapping class group* $\mathcal{PM}_n(M)$ of M is the group of isotopy classes of diffeomorphisms of $\widehat{M}$ which take X point-wise. The groups $\mathcal{M}_0(\widehat{M})$ and $\mathcal{PM}_0(\widehat{M})$ will be denoted by $\mathcal{M}(\widehat{M})$ and $\mathcal{PM}(\widehat{M})$ respectively.

Let M be a connected surface with boundary ∂M consisting of n connected components $V_1, \dots, V_n$. Regarding these components as punctures, we can identify the groups $\mathcal{M}(M)$ and $\mathcal{PM}(M)$ with $\mathcal{M}_n(\widehat{M})$ and $\mathcal{PM}_n(\widehat{M})$.

We recall the sets of generators of $\mathcal{M}(M)$ and $\mathcal{PM}(M)$ given in [B2, G] for orientable surfaces and in [KM] for nonorientable ones.

7.1. **Orientable case.** Suppose that M is orientable. Consider the following 3 types of diffeomorphisms of M:

(1) Let O be a reversing orientation diffeomorphism of M.

(2) Let $\alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i$ be the SCC shown in Figures 7.1a), where the bold points denote connected components of ∂M divided into two parts (positive and negative components). We will refer them as SCCs of configuration $\mathcal{C}$. Denote by t_{α_i}, t_{β_i}, t_{γ_i}, t_{δ_j}, t_{ϵ_i} the corresponding Dehn twists.

(3) For every pair $i < j = 1, \ldots, n$ let σ_{ij} be a SCC that separates M into two connected components so that one of which is a sphere S with 3 holes whose boundary components are σ_{ij} and the connected components V_i and V_j of ∂M, see Figure 7.1b). Let b_{ij} be a diffeomorphism of M with support in S which permutes boundary components V_i and V_j and preserves all others. Evidently, b_{ij}^2 is a Dehn twist $t_{\sigma_{ij}}$ along σ_{ij}.

Theorem 7.2 ([B2, G])**.** *The group $\mathcal{M}(M)$ is generated by*

(i) $\{O, b_{ij} : i, j = 1, \ldots, n\}$ *if* $g = 0$*;*

(ii) $\{t_l, O, b_{ij} : l \in \mathcal{C}, i, j = 1, \ldots, n\}$ *if* $g \geq 1$*.*

The group $\mathcal{PM}(M)$ is generated by

(i) $\{O, b_{ij}^2 = t_{\sigma_{ij}} : i, j = 1, \ldots, n\}$ *if* $g = 0$*;*

(ii) $\{t_l, O : l \in \mathcal{C}, i, j = 1, \ldots, n\}$ *if* $g \geq 1$*.*

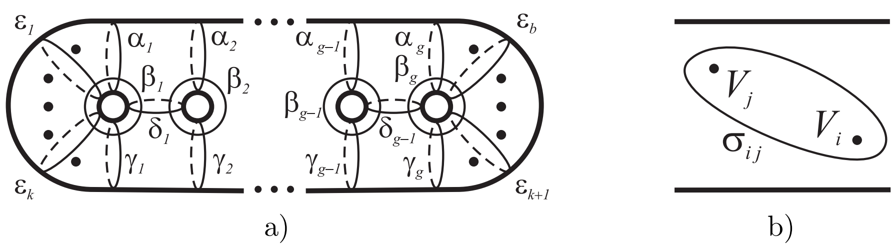


Figure 7.1. The configuration $\mathcal{C}$. Orientable case.

7.3. **Generators for $\mathcal{M}(M)$. Non-orientable case.** Suppose that M is non-orientable of genus g, see Figure 7.1, where the interiors of the shaded disks are removed and then the antipodal points on each boundary component are to be identified.

Consider the following 4 types of diffeomorphisms of M:

(1) Let y be a crosscap slide of M. If $g \geq 3$, then we additionally assume that y^2 is a Dehn twist along a two-sided separating SCC both components of whose complement are non-orientable.

(2) and (3) Similarly to the oriented case we define the configuration $\mathcal{C}$ of SCCs $\alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i$ shown in Figure 7.1, SCCs σ_{ij}, the corresponding Dehn twists and diffeomorphisms b_{ij}.

(4) Let ν_i denotes the boundary slide obtaining by sliding the boundary component V_i along the loop μ if g is odd and along μ_1 if g is even, see

Figure 7.2. Also if g is even, denote by ω_i the boundary slide obtaining by sliding V_i once along the loop μ_2.

Theorem 7.4 ([KM]). *The group $\mathcal{M}(M)$ is generated by*

(i) $\{\nu_k, b_{ij} : i, j, k = 1, \ldots, n, i < j\}$ *if* $g = 1$*;*
(ii) $\{t_{\beta_0}, y, \nu_k, b_{ij} : i, j, k = 1, \ldots, n, i < j\}$ *if* $g = 2$*;*
(iii) $\{t_l, y, \nu_k, b_{ij} : l \in \mathcal{C}, i, j, k = 1, \ldots, n, i < j\}$ *if* $g \geq 3$ *is odd;*
(iv) $\{t_l, y, \nu_k, \omega_k, b_{ij} : l \in \mathcal{C}, i, j, k = 1, \ldots, n, i < j\}$ *if* $g \geq 4$ *is even.*

Replacing every b_{ij} by $b_{ij}^2 = t_{\sigma_{ij}}$ we obtain generators for $\mathcal{PM}(M)$.

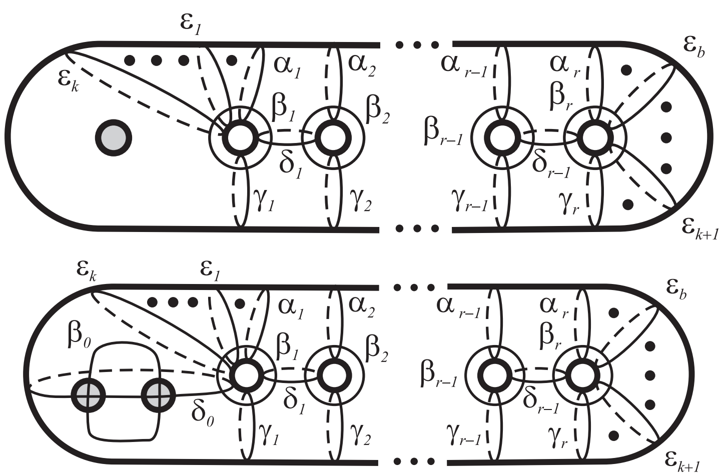


FIGURE 7.1. The configuration $\mathcal{C}$ for $g = 2r + 1$ and $g = 2r + 2$. Non-orientable case

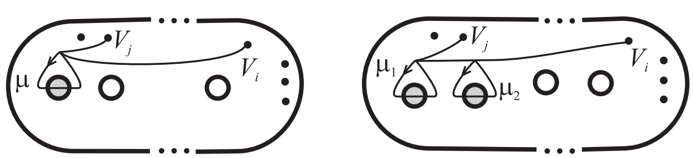


FIGURE 7.2. Boundary slides for $g = 2r + 1$ and $g = 2r + 2$.

7.5. **Generators of $\mathcal{M}(M)$ for canonical Morse mapping.** Given a Morse mapping f, denote by $\mathcal{M}_f(M)$ the subgroup of $\mathcal{M}(M)$ consisting of diffeomorphisms that preserve the sets of f-positive and f-negative components of ∂M. Evidently, $\mathcal{A}(f) \subset \mathcal{M}_f(M)$.

Lemma 7.6. *Let $f : M \to P$ be a canonical Morse mapping. In the case $P = S^1$ assume that M is orientable. Then there is a "canonical" set of generators for $\mathcal{M}_f(M)$ such that*

(i) *for the case* $P = \mathbb{R}^1$ *all of them are* f*-admissible, i.e.* $\mathcal{A}(f) = \mathcal{M}_f(M)$*, whence Main Theorem holds for this case;*
(ii) *for* $P = S^1$ *(and orientable* M*) all but one of them are also* f*-admissible.*

Remark 7.7. Recall that we do not give the proof of Main Theorem (by the new method) for the case M is non-orientable and $P = S^1$. Therefore we also do not consider this case in Lemma 7.6 since it is more complicated and due to the length of the paper, see also the last paragraph of this section.

Proof. Let f be a canonical Morse mapping. We will construct a set of generators for $\mathcal{M}(M)$ described in Theorems 7.2 and 7.4 such that their f-admissibility is rather evident.

First suppose M is orientable and embedded in $\mathbb{R}^3$ as it is shown in Figure 4.1. Then the canonical Morse mapping f is just the projection onto the vertical line.

(1) Let O be a diffeomorphism of M that is a symmetry with respect to the plane of this sheet. Then O reverses orientation of M and preserves f, i.e. $f = f \circ O$. Thus O is f-admissible.

(2) Comparing Figures 4.1 and 7.1 we see that α_i and γ_i are regular components of regular level-sets of f, whence the Dehn twists t_{α_i} and t_{γ_i} are admissible. In Figure 7.1 an f-admissibility of twists t_{β_i}, t_{δ_i} and t_{ϵ_i} is shown.

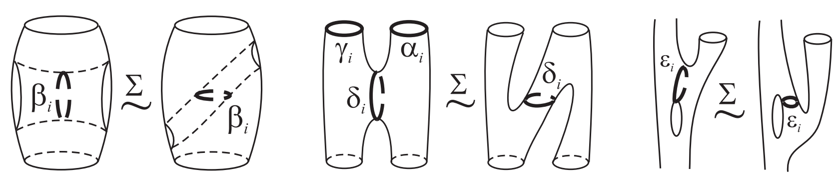


FIGURE 7.1. f-admissibility of configuration $\mathcal{C}$

(3) Let V_i and V_j be two f-positive components of ∂M. Then f is Σ-homotopic to a Morse mapping f_1 such that *KR*-graph Γ_{f_1} of f_1 includes a subgraph Γ_1 shown in Figure 7.2a). Let σ_{ij} be a SCC corresponding to a point $s \in \Gamma_1$. Then there exists a diffeomorphism b_{ij} of M_1 that exchanges V_i and V_j, preserves f_1 and b_{ij}^2 is a Dehn twist along σ_{ij}. Then b_{ij} and σ_{ij} are f-admissible.

Now let V_i be f-positive and V_j be f-negative. In this case a diffeomorphism b_{ij} permuting V_i and V_j is not f-admissible, since it does not preserve the sets of f-positive and f-negative boundary components. Nevertheless we will now show that its square $b_{ij}^2 = t_{\sigma_{ij}}$ is f-admissible. Consider two cases.

(a) Suppose that f has at least one critical point of index either 0 or 2 or a boundary component different from V_i and V_j. Then f is Σ-homotopic to a Morse mapping f_1 whose *KR*-graph Γ_{f_2} includes a subgraph Γ_2 shown

in Figure 7.2b). Then we define σ_{ij} to be a SCC corresponding to a point $s \in \Gamma_2$. Hence σ_{ij} is f-admissible.

(b) Otherwise, f has no local extremes and $\partial M = V_1 \cup V_2$. Let σ_{12} be a SCC that intersects every γ_i but no other SCCs of configuration $\mathcal{C}$, separates M in two components M_1 and M_2 such that M_1 is disk with two holes V_1 and V_2, see Figure 7.2c).

We claim that σ_{12} is not f-admissible. Otherwise the restriction of f to M_2 must have extremes, which could be taken only on boundary components different from V_1 and V_2 or at critical points of indexes 0 and 2. But all of them are absent on M_2.

Nevertheless, it is well-known that a Dehn twist $t_{\sigma_{12}}$ is a product of Dehn twists along SCCs of configuration $\mathcal{C}$ except for γ_i. Hence a Dehn twist $t_{\sigma_{12}}$ is f-admissible.

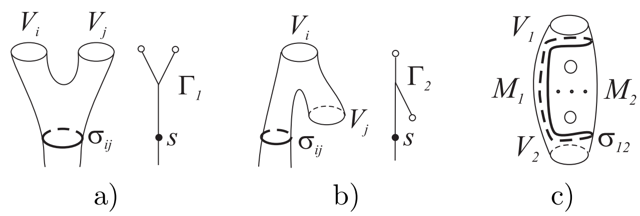


FIGURE 7.2. f-admissibility of b_{ij} and σ_{ij}

Suppose that M is non-orientable of genus g (see Figure 7.1) and let f be a canonical Morse mapping as in Figure 4.1. Again we define the generators of $\mathcal{M}(M)$ associated with f.

(1) For the case $g \geq 2$ we will now define an f-admissible crosscap slide. If g is odd then, Γ_f has an edge e with vertices of degree 2. Otherwise, f is Σ-homotopic to a Morse function f_1 whose KR-graph has such an edge, see Figure 4.1d). Then by Lemma 6.9, there exists a crosscap slide y such that $f = f \circ y$ or $f_1 = f_1 \circ y$ in the second case. Hence y is f-admissible.

Definition and f-admissibility of generators of types (2) and (3) are similar to the orientable case. We need to verify the admissibility of β_0 and δ_0 for the case $g = 2r \geq 2$.

Let N be a neighborhood of e defined just above containing no vertices of Γ_f but ∂e. Then $K = f_\Gamma^{-1}(N) \subset M$ is a Klein bottle with two holes. Let $p : T \to K$. Then T is a torus with four holes. We can assume that the function $\widetilde{f} = f \circ p : T \to \mathbb{R}$ coincides with one defined in Section 6.8, see Figure 6.1. Since β_0 and δ_0 are two sided, their inverse images $\widetilde{\beta}_0 = p^{-1}(\beta_0)$ and $\widetilde{\delta}_0 = p^{-1}(\delta_0)$ in T consist of pair of disjoint SCC. They are shown in Figure 7.3a).

It is shown in Figure 7.3b) that $\widetilde{\beta}_0$ is a regular level-set of $\widetilde{f}$. This figure also shows a symmetrical Σ-homotopy of $\widetilde{f}$ fixed near ∂T which makes $\widetilde{\delta}_0$ a regular level-set. Hence $\widetilde{\beta}_0$ and $\widetilde{\delta}_0$ are $\widetilde{f}$-admissible, whence β_0 and δ_0 are f-admissible.

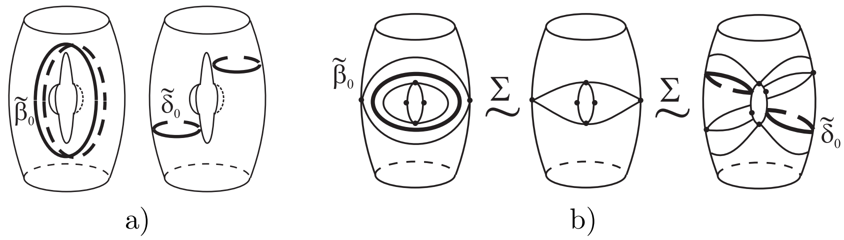


FIGURE 7.3. f-admissibility of β_0 and δ_0

(4) It remains to construct f-admissible boundary slides ν_i and ω_i. Let V_i be a connected component of ∂M and $z_i \in \Gamma_f$ be the corresponding $\circ$-vertex.

First suppose that g is odd, so Γ_f has a unique vertex x of degree 2. Then f is Σ-homotopic to a Morse function f_1 such that z_i and x will be the vertices of same edge, see Figure 7.4 for the cases when z_i is f-negative or f-positive. Then by Lemma 6.7, there exists a boundary slide ν_i of V_i preserves f_1. Whence ν_i is f-admissible.

If g is even, then Γ_f has two vertices x_1 and x_2 of degrees 2. As in the previous case we define f-admissible boundary slices ν_i for V_i and x_1, and ω_i for V_i and x_2.

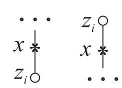


FIGURE 7.4.

Consider now the case $P = S^1$. Let $c \in S^1$ be a regular value of f and $\alpha_1 = f^{-1}(c)$ such that the restriction of f to $M \setminus \alpha_1$ is a canonical Morse function to $S^1 \setminus c$.

Suppose that M is orientable. Then the definition configuration of the $\mathcal{C}$ associated with f is shown in Figure 7.5, where f is the "projection" to β_1. Similarly to the previous case we can define a diffeomorphism O, Dehn twists along the SCCs of configuration $\mathcal{C}$, and permutations of boundary components b_{ij}. The same arguments as in the case $P = \mathbb{R}$ show that all of them are admissible, except for β_1, since f and $f \circ t_{\beta_1}$ are not even homotopic.

If M is non-orientable, then the surface $M \setminus \alpha_1$ can be orientable or non-orientable as well. We do not consider this case, see 7.7. $\square$

8. PROOF OF MAIN THEOREM.

The case $P = \mathbb{R}$ is proved in statement (i) of Lemma 7.6. Before processing with the case $P = S^1$ we recall the definition of Torelli group and its generators.

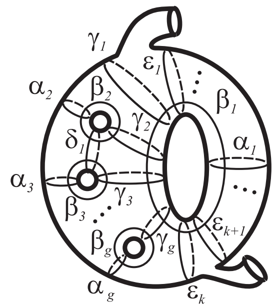


FIGURE 7.5. Configuration $\mathcal{C}$ if M is orientable and $P = S^1$

8.1. **Torelli group $\mathcal{T}(M)$.** Let M be a closed orientable surface. Then the Torelli group of M is a subgroup $\mathcal{T}(M)$ of $\mathcal{PM}(M) = \mathcal{M}(M)$ consisting of diffeomorphism of M trivially acting on the homology group $H_1(M)$. Evidently, $\mathcal{T}(M)$ is a normal subgroup in $\mathcal{PM}(M)$.

Suppose now that $\partial M \neq \varnothing$. Let us glue every connected component of ∂M by 2-disk and denote the obtained closed surface by $\widehat{M}$. Then we obtain an epimorphism $j : \mathcal{PM}(M) \to \mathcal{PM}(\widehat{M})$ induced by the inclusion $M \subset \widehat{M}$, see [B2]. Define the Torelli group $\mathcal{T}(M) \subset \mathcal{PM}(M)$ of M to be the inverse image $j^{-1}(\mathcal{T}(\widehat{M}))$.

The following theorem describes the generators of $\ker j$.

Theorem 8.2 ([B1, B2])**.** *Let α_i and β_i be the curves of configuration $\mathcal{C}$ on M. For every component V_j of ∂M let α_{ik} (β_{ik}) be an SCC which together with α_i (β_i) bounds in M a cylinder with a hole V_i. Then the kernel of j is generated by the following diffeomorphisms: $s_{ik} = \alpha_i \circ \alpha^{1}_{ik}$ and $r_{ik} = \beta_i \circ \beta^{1}_{ik}$.*

Theorem 8.3 ([B3, P, J, MG])**.** *The Torelli group $\mathcal{T}(M)$ of M is generated by the following types of diffeomorphisms:*

(a) *Dehn twists along SCC separating M (if $g = 2$ then these diffeomorphisms generate all the group $\mathcal{T}(M)$,* [MG]*);*

(b) *products of Dehn twists of the form $t_{\gamma_1} \circ t^{-1}_{\gamma_2}$, where the SCCs γ_1 and γ_2 are oriented, disjoint, and homologous.*

Proof. This theorem was proved for closed surfaces [P] and surfaces with one boundary component [J]. In fact it holds for arbitrary oriented surfaces.

Let $t \in \mathcal{T}(M)$. Since $\widehat{M}$ is closed, we have that $j(t)$ is generated by diffeomorphisms of types (a) and (b). Notice that we can choose the corresponding curves so that they belong to M, whence $j(t)$ yields some diffeomorphism t_1 of surf such that $t_1^{-1} \circ t \in \ker j$. By Theorem 8.2, this diffeomorphism is also generated by diffeomorphisms s_{ik} and r_{ik} which evidently are of type (b). $\square$

8.4. **Proof of Main Theorem for orientable M and $P = S^1$.** It suffices to establish the following statement using the notations of Lemma 7.6.

Proposition 8.5. *Let $h \in \mathcal{M}_f(M)$ be a diffeomorphism such that the Morse mappings f and $f \circ h : M \to S^1$ are homotopic. Then h is isotopic to a product of diffeomorphisms of the form $p \circ c \circ t$, where*

(1) *p is generated by O and those b_{ij} that belong $\mathcal{M}_f(M)$;*
(2) *c is generated by Dehn twists along the SCCs of configuration $\mathcal{C}$ but t_{β_1};*
(3) *$t \in \mathcal{T}(M)$.*

Diffeomorphisms of types (1)-(3) *are f-admissible, whence so is h.*

Proof. Evidently that h can be represented as a product $p \circ h_1$, where $h_1 \in \mathcal{PM}(M)$ and preserves orientation of M and p is of type (1). Then by Theorem 7.2, that h_1 is generated by the Dehn twists along the curves of configuration $\mathcal{C}$.

Notice that f and $f \circ h_1$ are homotopic. This condition will allow us to remove t_{β_1} from the generators of h_1 and replace this twist by diffeomorphisms of type (3).

Lemma 8.6. *Let h_1 be a diffeomorphism of M generated by the Dehn twists along the SCCs of configuration $\mathcal{C}$ and such that f and $f \circ h_1$ are homotopic. Then there exists an f-admissible diffeomorphism c generated by the Dehn twists along the SCCs of configuration $\mathcal{C}$ except for t_{β_1} such that the diffeomorphism $t = c^{-1} \circ h_1$ belongs to $\mathcal{T}(M)$.*

Hence it remains to establish that every diffeomorphism $t \in \mathcal{T}(M)$ is f-admissible. By Theorem 8.3 it suffices to prove this for diffeomorphism of types (a) and (b).

Theorem 8.7. *Let $f : M \to S^1$ be a Morse mapping.*

(i) *Let $\gamma \subset M$ be an SCC and t_γ be a Dehn twist along γ. Then t_γ is f-admissible iff the restriction $f|_\gamma$ is null-homotopic. If γ separates M, then $f|_\gamma$ is null-homotopic, whence every diffeomorphism of type* (a) *is f-admissible.*

(ii) *Every diffeomorphism of type* (b) *is f-admissible.*

Thus in order to complete our proposition, and therefore Main Theorem, it remains to prove Theorem 8.7 (sections 12 and 13) and Lemma 8.6 (section 14).

9. Symplectic group

For the proof of Lemma 8.6 we need a description of generators of stabilizers in symplectic group $Sp_{2g}(\mathbb{Z})$. The representation of the group $Sp_{2g}(\mathbb{Z})$ is given in [B3]. We will also use the ideas from [OM].

Let $\mathbb{Z}^{2g}$ be a free $2g$-module with basis

$$\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \tag{9.1}$$

I be the unity $g \times g$-matrix, and e_{ij} be a $g \times g$-matrix, whose (i, j)-element (the intersection of i-th row and j-th column) is equal to 1 and all other entries are zeros.

Let also ω be a skew-symmetric 2-form whose matrix in the basis (9.1) is the following:

$$\left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right). \tag{9.2}$$

Thus $\omega(\alpha_i, \beta_i) = 1$ and $\omega(\alpha_i, \alpha_j) = \omega(\beta_i, \beta_j) = \omega(\alpha_i, \beta_j) = 0$ for $i, j = 1, \dots, g$. The group of all linear isomorphisms of $\mathbb{Z}^{2g}$ preserving ω is denoted by $Sp_{2g}(\mathbb{Z})$ and is called *symplectic.*

9.1. **Transvections.** For every $\gamma \in \mathbb{Z}^{2g}$ the following automorphism t_γ of $\mathbb{Z}^{2g}$ defined by the formula:

$$t_\gamma(x) = \omega(\gamma, x) \cdot \gamma + x, \qquad \forall x \in \mathbb{Z}^{2g}. \tag{9.1}$$

is called the *transvection* along γ. It is easy to see that $t_\gamma \in Sp_{2g}(\mathbb{Z})$ and

$$t_\gamma^{-1}(x) = -\omega(\gamma, x) \cdot \gamma + x, \qquad \forall x \in \mathbb{Z}^{2g}.$$

Define the following elements of $Sp_{2g}(\mathbb{Z})$:

$$\begin{gathered} \mu_{ij} = t_{\alpha_i} \circ t_{\alpha_j} \circ t^{-1}_{\alpha_i+\alpha_j}, \qquad \eta_{ij} = t_{\beta_i} \circ t_{\beta_j} \circ t^{-1}_{\beta_i+\beta_j}, \\ \nu_{ij} = t_{\alpha_i} \circ t_{\beta_j} \circ t^{-1}_{\alpha_i+\beta_j}. \end{gathered} \tag{9.2}$$

Lemma 9.2. *The following formulas hold true for $i \neq j = 1 \dots g$:*

$$t_{\alpha_i} = \left\| \begin{array}{c|c} I & e_{ii} \\ \hline 0 & I \end{array} \right\|, \quad t_{\beta_i} = \left\| \begin{array}{c|c} I & 0 \\ \hline -e_{ii} & I \end{array} \right\|, \quad t_{\alpha_i+\beta_j} = \left\| \begin{array}{c|c} I - e_{ij} & e_{ii} \\ \hline -e_{jj} & I + e_{ji} \end{array} \right\|.$$

$$t_{\alpha_i+\alpha_j} = \left\| \begin{array}{c|c} I & e_{ii} + e_{jj} + e_{ij} + e_{ji} \\ \hline 0 & I \end{array} \right\|,$$

$$t_{\beta_i+\beta_j} = \left\| \begin{array}{c|c} I & 0 \\ \hline -e_{ii} - e_{jj} - e_{ij} - e_{ji} & I \end{array} \right\|,$$

$$\mu_{ij} = \left\| \begin{array}{c|c} I & -e_{ij} - e_{ji} \\ \hline 0 & I \end{array} \right\|, \quad \eta_{ij} = \left\| \begin{array}{c|c} I & 0 \\ \hline e_{ij} + e_{ji} & I \end{array} \right\|$$

$$\nu_{ij} = \left\| \begin{array}{c|c} I + e_{ij} & 0 \\ \hline 0 & I - e_{ji} \end{array} \right\|.$$

Moreover, the matrices t_{α_i}, t_{β_i}, μ_{ij}, η_{ij}, and ν_{ij}, $(i \neq j = 1 \dots g)$ generate $Sp_{2g}(\mathbb{Z})$.

Proof. The lemma can be established by direct calculations. The fact that these matrices generate $Sp_{2g}(\mathbb{Z})$ can be easily deduced from [OM, Ch. 2, §2.2.] or [B3]. □

For each $x \in \mathbb{Z}^{2g}$ denote by $T(x)$ the subgroup in $Sp_{2g}(\mathbb{Z})$ generated by transvections along elements of $\mathbb{Z}^{2g}$ that are ω-orthogonal to x, i.e.

$$T(x) = \langle t_\gamma \mid \gamma \in \mathbb{Z}^{2g}, \omega(\gamma, x) = 0 \rangle. \tag{9.1}$$

Let also $\mathrm{St}(x)$ be the stabilizer of x in $Sp_{2g}(\mathbb{Z})$, i.e.

$$\mathrm{St}(x) = \{h \in Sp_{2g}(\mathbb{Z}) \mid h(x) = x\}.$$

It easily follows from (9.1) that $T(x) \subset \mathrm{St}(x)$.

Proposition 9.3. $T(\alpha_1) = \mathrm{St}(\alpha_1)$. *Moreover, this group is generated by the following matrices:*

$$(9.1) \quad t_{\alpha_i},\ t_{\beta_i},\ \mu_{ij},\ \eta_{ij},\ \nu_{ij}, \qquad \textit{except for } t_{\beta_1},\ \eta_{1i} = \eta_{i1} \textit{ and } \nu_{i1},\ (i \neq j = 1, \dots, g).$$

Proof. Evidently, the matrices (9.1) belong to $T(\alpha_1)$. Let $h \in \mathrm{St}(\alpha_1)$. We will show that h is generated by (9.1). The proof consists of two steps.

Step 1. We will find an element $h_1 \in Sp_{2g}(\mathbb{Z})$ such that $h \cdot h_1^{-1}$ is generated by (9.1) and $h_1(\beta_1) = \beta_1$. Let

$$h(\beta_1) = a_1\,\alpha_1 + b_1\,\beta_1 + a_2\,\alpha_2 + b_2\,\beta_2 + \dots,$$

for some $a_i, b_i \in \mathbb{Z}, (i = 1 \dots, g)$. Since h preserves the form ω and $h(\alpha_1) = \alpha_1$, we get

$$b_1 = \omega\,(\alpha_1, h(\beta_1)) = \omega\,(h(\alpha_1), h(\beta_1)) = \omega\,(\alpha_1, \beta_1) = 1.$$

Consider now the effect of action of μ_{1j} and ν_{1j} on $h(\beta_1)$, $j = 2 \dots g$. Let $t \in \mathbb{Z}$. Then it is easy to verify that for $j > 1$ we have:

$$(\mu_{1j})^t \circ h(\beta_1) = (a_1 - tb_j)\,\alpha_1 + \beta_1 + \dots + (a_j - t)\,\alpha_j + b_j\,\beta_j + \dots,$$
$$(\nu_{1j})^t \circ h(\beta_1) = (a_1 + ta_j)\,\alpha_1 + \beta_1 + \dots + a_j\,\alpha_j + (b_j - t)\,\beta_j + \dots,$$

where the coefficients at other basis elements are not changed.

Define now $h_1 \in \mathbb{Z}^{2g}$ by the formula:

$$h_1 = (t_{\alpha_1})^{-a'} \cdot \prod_{j=2}^{g} (\nu_{1j})^{b_j} \cdot \prod_{i=2}^{g} (\mu_{1j})^{a_j} \cdot h,$$

where

$$a' = a_1 - \sum_{j=2}^{g} a_j b_j.$$

We claim that $h_1(\beta_1) = \beta_1$.

Indeed, the product of μ_{1j} reduces the coefficients at α_j and the product of ν_{1j} reduces the coefficients at β_j for every $j = 2 \dots g$. This also makes the coefficient at α_1 equal to a'. Since

$$t_{\alpha_1}(\alpha_1) = \alpha_1 \qquad \text{and} \qquad (t_{\alpha_1})^t(\beta_1) = (a_1 + t)\,\alpha_1 + \beta_1,$$

we obtain that the multiple $(t_{\alpha_1})^{-a'}$ reduces this coefficient.

Step 2. Consider the following submodules of $\mathbb{Z}^{2g}$:

$$P = \langle \alpha_1, \beta_1 \rangle \qquad \text{and} \qquad Q = \langle \alpha_i, \beta_i \mid i = 2 \dots g \rangle.$$

They are orthogonal with respect to the form ω and $h_1|_P = \mathrm{id}$. Since h_1 preserves ω, it follows that $h_1(Q) = Q$. Thus h_1 can be regarded as an element of the group $Sp_{2g-2}(\mathbb{Z}) \subset Sp_{2g}(\mathbb{Z})$ consisting of isomorphisms that are identity on P.

By Lemma 9.2 the group $Sp_{2g-2}(\mathbb{Z})$ is generated by matrices (9.1) for $i \neq j = 2 \dots g$. In particular, they generate h_1. $\square$

10. Minimal Morse maps.

For the proof of Theorem 8.7 we need the notion of minimal Morse mapping. Let M be a compact surface orientable or not. We say that a Morse map $f : M \to P$ is *minimal* if the number $c_0(f) + c_1(f) + c_2(f)$ of critical points of f is minimal among all possible Morse maps $M \to P$ having the same sets of positive and negative boundary components as f. Let b_+ and b_- be the number of f-positive and f-negative boundary components of M. The following lemma is easy to prove:

Lemma 10.1. *A Morse mapping $f : M \to P$ is minimal iff for every connected component X of M the restriction $f|_X$ is minimal. A Morse function $f : M \to \mathbb{R}^1$ on a* connected *surface M is minimal if and only if the following two relations hold true*

$$(10.1) \qquad c_0(f) = \begin{cases} 1, & \text{if } b_- = 0, \\ 0, & \text{if } b_- > 0, \end{cases} \qquad c_2(f) = \begin{cases} 1, & \text{if } b_+ = 0, \\ 0, & \text{if } b_+ > 0. \end{cases}$$

Let $f : M \to S^1$ be a Morse mapping which is not null-homotopic. Then f is minimal iff $c_0(f) = c_2(f) = 0$. $\square$

We admit now that M may be not connected. Let $f : M \to [0, 1]$ be a Morse function such that $\frac{1}{2} \in [0, 1]$ is its regular value. Denote

$$V_0 = f^{-1}[0, 1/2], \quad V_1 = f^{-1}[1/2, 1].$$

$$B_0 = f^{-1}(0), \quad B_1 = f^{-1}(1), \quad Z = f^{-1}(1/2)$$

Lemma 10.2. *Suppose that*

1. *B_0, B_1 and Z are nonempty, the union $B_0 \cup B_1$ is included in ∂M and intersects every connected component of M;*
2. *the restriction $f|_{V_i}$ is a* minimal *Morse function for $i = 0, 1$;*
3. *for every connected component X of M such that $X \cap Z \neq \varnothing$ we have $X \cap B_i \neq \varnothing$ for both $i = 0, 1$.*

Then f is a minimal Morse function on M.

Proof. Let X be a component of M. We will show that $f|_X$ is a minimal Morse function. Denote $X_i = X \cap V_i$ $(i = 0, 1)$.

If $X \cap Z = \varnothing$, then X is a connected component of one of the sets either V_0 or V_1. Then the restriction of f onto X is minimal.

Suppose that $X \cap Z \neq \varnothing$. Then $X \cap B_i \neq \varnothing$ for $i = 0, 1$ by (3). Evidently, the components of the intersection $X \cap Z \neq \varnothing$ are negative for the restriction $f|_{X_1}$ and positive for the restriction $f|_{X_0}$. Therefore, by Lemma 10.1, we have

$$(10.1) \qquad c_2(f|_{X_0}) = c_0(f|_{X_1}) = 0.$$

Similarly, the intersection $X \cap B_0$ (resp. $X \cap B_1$) consists of some negative (resp. positive) components of $f|_X$ and $f|_{X_0}$ (resp. $f|_{X_1}$). Then from Lemma 10.1, we also get

$$c_0(f|_{X_0}) = c_2(f|_{X_1}) = 0.$$

Combining this with (10.1), we obtain

$$c_i(f|_X) = c_i(f|_{X_0}) + c_i(f|_{X_1}) = 0, \quad i = 0, 2.$$

Whence by Lemma 10.1 $f|_X$ is minimal. □

11. Minimization of intersections with a level-set

Let M be a compact surface (orientable or not), $f : M \to S^1$ be a Morse mapping, and $\gamma_1, \dots, \gamma_m \subset M$ be disjoint SCCs.

Lemma 11.1. *f is Σ-homotopic to a Morse mapping g such for some level-set L of g and for every $i = 1, \dots, m$ the curve γ_i does not pass through the critical points of g and*

(i) *if the restriction $f|_{\gamma_i}$ is not null-homotopic, then γ_i transversely intersects every level-set of g;*
(ii) *otherwise $\gamma_i \cap L = \varnothing$.*

Proof. Let $c \in S^1$ be a regular value of f. Set

$$\Gamma = \mathop{\cup}_{i=1}^{m} \gamma_i, \qquad n = \#[f^{-1}(c) \cap \Gamma], \qquad \text{and} \qquad d = \sum_{i=1}^{m} |\deg f|_{\gamma_i}|.$$

Then $\#[f^{-1}(c) \cap \gamma_i] \geq \deg f|_{\gamma_i}$ for $i = 0, 1$, whence $n \geq d$. Moreover, $n = d$ if and only if $\#[f^{-1}(c) \cap \gamma_i] = \deg f|_{\gamma_i}$.

Claim 11.2. *Suppose that $n > d$. Then f is Σ-homotopic to a Morse map f_1 such that $\#[f_1^{-1}(c_1) \cap \Gamma] < n$ for some regular value c_1 of f_1.*

Proof. We will exploit the notations and the construction of Section 3. Cutting M along $f^{-1}(c)$ we obtain the surface $\widetilde{M}$ and the Morse function $\widetilde{f} : \widetilde{M} \to [0, 1]$. Let also $p : \widetilde{M} \to M$ be the factor-map, $B_i = \widetilde{f}^{-1}(i)$ for $i = 0, 1$, and $B = B_0 \cup B_1 = p^{-1}(f^{-1}(c))$.

Let $L = p^{-1}(\Gamma)$ and $l_1, \dots, l_k$ be the connected components of L. Then the intersection $l_j \cap B$ is either empty (whence l_j is an SCC) or consists of two points (whence l_j is a simple arc with ends in B). Let us divide L into four groups $L_\varnothing$, L_0, L_0^1, L^1 consisting of arcs that respectively do not intersect B, intersect only B_0, intersect both sets B_0 and B_1, and intersect only B_1. Thus $L = L_\varnothing \cup L_0 \cup L_0^1 \cup L^1$. Notice that $\#[L \cap B_0] = \#[L \cap B_1] = n$, $\#[L_0] = \#[L^1]$, and the sets L_0 and L^1 are non-empty if and only if $n > d$.

Let $Q_0^1 \subset \widetilde{M}$ be a union of those connected components of $\widetilde{M}$ which intersect both sets B_0 and B_1. Consider the set

$$G = Q_0^1 \cap (B_0 \cup L_0).$$

By definition, $G \cap (L_\varnothing \cup L^1) = \varnothing$. Then there exists a regular neighborhood W of G which does not intersect $L_\varnothing \cup L^1$ and such that the boundary $Z = \partial W$ transversely intersects every component of L^1_0 at a unique point. Hence, $Z \cap L = Z \cap L^1_0$. Evidently, Z separates $\widetilde{M}$ between B_0 and B_1. Moreover, $\#[Z \cap L^1_0] < n$.

We will now construct a Morse function $\widetilde{g} : \widetilde{M} \to [0,1]$ which coincides with $\widetilde{f}$ in some neighborhood of $B \cup \partial\widetilde{M}$, has critical type of $\widetilde{f}$, and such that $\widetilde{g}^{-1}(\frac{1}{2}) = Z$.

Let $\widetilde{g}_0 : V_0 \to [0, \frac{1}{2}]$ and $\widetilde{g}_1 : V_1 \to [\frac{1}{2}, 1]$ be two minimal Morse functions such that

$$\widetilde{g}_0^{-1}(0) = B_0, \qquad \widetilde{g}_0^{-1}(1/2) = \widetilde{g}_1^{-1}(1/2) = Z, \qquad \widetilde{g}_1^{-1}(1) = B_1,$$

and the Morse function $\widetilde{g} : \widetilde{M} \to [0,1]$ defined by $\widetilde{g}|_{V_i} = \widetilde{g}_i$, $(i = 0,1)$ is C^∞, has the same sets of positive and negative components as $\widetilde{f}$, and coincide with $\widetilde{f}$ in some neighborhood of $B \cup \partial\widetilde{M}$.

We claim that $\widetilde{g}$ is minimal. Indeed, let X be a component of $\widetilde{M}$ such that $X \cap Z \neq \varnothing$. Since $Z = \partial W \subset Q^1_0$, we obtain that $X \subset Q^1_0$. Denote $X_i = X \cap V_i$, then $X \cap B_i = X_i \cap B_i \neq \varnothing$, by the definition of Q^1_0. It follows from Lemma 10.2 that $\widetilde{g}$ is minimal.

Adding critical points to $\widetilde{g}$ outside of $B \cup Z$ we can change its critical type to the critical type of $\widetilde{f}$. Let us denote this new function by $\widetilde{f}_1$. Then $\widetilde{f}_1$ satisfies the statement of our claim.

Indeed, denote $c_1 = q(\frac{1}{2})$. By the case $P = \mathbb{R}^1$ of Main Theorem we obtain that $\widetilde{f} \overset{\Sigma}{\sim} \widetilde{f}_1$ with respect to some neighborhood of $B \cup \partial\widetilde{M}$. This Σ-homotopy induces a Σ-homotopy (with respect to $f^{-1}(c) \cup \partial M$) of f to a Morse mapping f_1 such that $\#[f_1^{-1}(c_1) \cap \Gamma] < n$. $\square$

We now proceed with the proof of Lemma 11.1. By Claim 11.2 we can assume that $n = d$. As noted above this is equivalent to the statement: $\#[f^{-1}(c) \cap \gamma_i] = \deg f|_{\gamma_i}$. In particular, if the restriction $f|_{\gamma_i}$ is null-homotopic, then $\#[f^{-1}(c) \cap \gamma_i] = 0$, i.e. $\gamma_i \cap f^{-1}(c) = \varnothing$, whence (ii) holds true.

Let us assume that l_i is given by an embedding $l_i : [0,1] \to \widetilde{M}$ so that $l_i \cap l_j = \varnothing$ for $j \neq i$. To establish (i) we prove that following claim:

Claim 11.3. *Suppose that $l_i(0) \in B_0$, $l_i(1) \in B_1$, and the intersection $l_i \cap B$ is transversal for each $i = 1, \ldots, k$. Then $\widetilde{f}$ is Σ-homotopic to a Morse function $\widetilde{g}$ such that l_i is transversal to level-sets of $\widetilde{g}$.*

It follows that a Σ-homotopy of this claim yields a Σ-homotopy $f \overset{\Sigma}{\sim} g$ with respect $f^{-1}(c)$ such that every γ_i is transversal to level-sets of g. This will complete Lemma 11.1.

Proof of Claim 11.3. We will construct a Morse function $\widetilde{f}_1$ and a gradient-like vector field F for $\widetilde{f}_1$ such that for every $i = 1, \ldots, m$ the arc l_i is a

trajectory of F. Then adding or canceling the proper number of pairs of critical points of $\widetilde{f}_1$ outside $\cup_i l_i$ we obtain a Morse function $\widetilde{g}$ having the critical type of $\widetilde{f}$ and such that F is a gradient-like for $\widetilde{g}$.

For every $i = 1, \ldots, m$ let $\phi_i : [0,1] \times [-1,1] \to M$ be a smooth embedding such that the image $V_i = \mathrm{Im}\phi_i$ is a neighborhood of l_i, $\phi_i(t,0) = l_i(t)$ for $t \in [0,1]$, $\phi^{-1}(B_s) = \{s\} \times [-1,1]$ for $s = 0,1$. Since l_i are mutually disjoint, we can assume that so are V_i. Denote $V = \cup_{i=1}^m V_i$ and define a function $\widetilde{g} : V \to [0,1]$ by the formula $\widetilde{g}(x) = p_2 \circ \phi_i^{-1}(x)$ for $x \in V_i$, where $p_2 : [0,1] \times [-1,1] \to [-1,1]$ is the natural projection.

Slightly changing $\widetilde{g}$ outside some neighborhood of $\cup_i l_i$ we can extend $\widetilde{g}$ over all of $\widetilde{M}$. Moreover, this extension may be assumed Morse whose positive and negative boundary components coincide with ones of $\widetilde{f}$ though the number of critical points of $\widetilde{g}$ and $\widetilde{f}$ may be different. Now we show how to change the critical type $K(\widetilde{g})$ of $\widetilde{g}$ to $K(\widetilde{f})$ by adding or canceling pairs of critical points outside of $\cup_i l_i$.

Recall that a vector field F on a manifold $\widetilde{M}$ is *gradient-like* for a function $\widetilde{f} : \widetilde{M} \to \mathbb{R}^1$ if $d\widetilde{f}(F)(x) > 0$ at each regular point x of $\widetilde{f}$.

Let Ψ be any gradient-like vector field for the function $\widetilde{g}$ on $\widetilde{M}$ and $\widetilde{\Phi}$ be the gradient vector field for the function p_2 on $[0,1] \times [-1,1]$, i.e. $\widetilde{\Phi}(s,t) = (0,1)$. Using ϕ_i we transfer $\widetilde{\Phi}$ to V_i. This gives us a vector field Φ on V such that l_i is a trajectory of Φ for $i = 1, \ldots, m$.

Finally, we glue Ψ and Φ. Let V' be a neighborhood of $\cup_i l_i$ such that $\overline{V'} \subset V$ and let $W = \widetilde{M} \setminus \overline{V'}$. Then $V \cup W = \widetilde{M}$.

Let $\mu_1, \mu_2 : \widetilde{M} \to [0,1]$ be a partition of unity corresponding to the open covering $\{V, W\}$ of $\widetilde{M}$, i.e. supp $\mu_1 \subset V$, supp $\mu_2 \subset W$, and $\mu_1 + \mu_2 \equiv 1$. Define a vector field F on $\widetilde{M}$ by the formula

$$F(x) = \mu_1(x) \cdot \Phi(x) + \mu_2(x) \cdot \Psi(x), \qquad x \in \widetilde{M}.$$

Evidently, F is gradient-like for $\widetilde{g}$ and coincides with Φ near $\cup_i l_i$. In particular, every l_i is a trajectory of F, whence l_i transversely intersects level-sets of $\widetilde{g}$.

It remains to show that $\widetilde{g}$ can be changed outside $\cup_i l_i$ to have critical type of $\widetilde{f}$. First we show how to make $\widetilde{g}$ a minimal Morse function.

Suppose that $\widetilde{g}$ has a critical point z_0 either of index 0 or 2. Since the sets of positive and negative boundary components of $\widetilde{g}$ are non-empty, there exists a critical points z_1 of index 1 and a trajectory ω of F with ends at z_0 and z_1. This trajectory does not intersect $\cup l_i$. Hence $\widetilde{g}$ can be changed in some neighborhood N of ω to have no critical points in N (see [HM, MJ1]). Thus the number of critical points is reduced. By the similar procedure we can add pairs of critical points outside of $\cup_i l_i$. Therefore we can change the critical type $K(\widetilde{g})$ of $\widetilde{g}$ to $K(\widetilde{f})$ leaving l_i transversal to level-sets of $\widetilde{g}$. $\square$

12. Proof of (i) of Theorem 8.7

Let $\gamma \subset M$ be a simple closed curve and t_γ be a Dehn twist along γ.

Necessity. Suppose that t_γ is f-admissible. Then f and $f \circ t_\gamma$ are homotopic. We should show that $\deg f|_\gamma = 0$. We can assume that there is a regular value c of f such that $\alpha = f^{-1}(c)$ is an SCC. Denote $\alpha' = t_\gamma(\alpha)$.

Since f and $f \circ t_\gamma$ are homotopic, we obtain from the last paragraph of Section 3.3 that $[\alpha'] = [\alpha]$ in $H_1(M, \partial M)$, i.e. t_γ fixes $[\alpha]$. Then by Eq. (9.1) for the action of Dehn twist in $H_1(M, \partial M)$ we get

$$[\alpha] = t_\gamma([\alpha]) = \omega\left([\gamma], [\alpha]\right) \cdot [\gamma] + [\alpha] = \deg f|_\gamma \cdot [\gamma] + [\alpha],$$

whence $\deg f|_\gamma = 0$.

Sufficiency. Suppose that $f|_\gamma$ is null-homotopic. By Lemma 11.1, f is Σ-homotopic to a Morse mapping g such that $g^{-1}(c) \cap \gamma = \varnothing$ for some regular value c of g. We apply now the construction of Section 3. Cutting M along $g^{-1}(c)$ we obtain a surface $\widetilde{M} = \widetilde{M}(g, c)$, a Morse function $\widetilde{g} : \widetilde{M} \to [0, 1]$, and an SCC $\widetilde{\gamma} \subset \widetilde{M}$ corresponding to γ. By the case $P = \mathbb{R}^1$ of Main Theorem, $t_{\widetilde{\gamma}}$ is $\widetilde{g}$-admissible. Then t_γ is g-admissible and therefore is f-admissible. $\square$

13. Proof of (ii) of Theorem 8.7

Let $f : M \to S^1$ be a Morse mapping, γ_1, γ_2 be disjoint oriented homologous simple closed curves in M, and $t = t_{\gamma_1} \circ t_{\gamma_2}^{-1}$ be the product of Dehn twists along these curves. We must prove that t is f-admissible.

Since these curves are homologous, it follows that the restrictions of f to them are homotopic. If these restrictions are null-homotopic, then by the case (i) of this theorem t is f-admissible. Therefore we will assume that $f|_{\gamma_1} \not\simeq 0$.

By Lemma 11.1 we can also assume that γ_i transversely intersects each level-set of g. Then the statement (ii) of Theorem 8.7 is a direct corollary of the following lemma:

Lemma 13.1. *Let $f : M \to S^1$ be a Morse mapping, γ_1, γ_2 be two disjoint homologous SCCs in M, and $t = t_{\gamma_1} \circ t_{\gamma_2}^{-1}$. Suppose that both of γ_i transversely intersect every level-set of f. Then $f \overset{\Sigma}{\sim} f \circ t$.*

Proof. Let $X \subset M$ be the closure of one of the connected components of $M \backslash (\gamma_1 \cup \gamma_2)$ bounded by the curves γ_1 and γ_2. Since γ_k $(k = 1, 2)$ transversely intersects level-sets of g, there exists an embedding ϕ_k of $S^1 \times [-2, 2]$ onto some neighborhood N_k of γ_k such that

$$\phi_k(S^1 \times \{0\}) = \gamma_k, \qquad \phi_k(S^1 \times [0, 2]) \subset X, \tag{13.1}$$

and the following diagram is commutative:

$$
\begin{array}{ccc}
S^1 \times [-2,2] & \xrightarrow{\phi_k} & N_k \subset M \\
{\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
S^1 & \xrightarrow{\sigma} & S^1
\end{array}
\tag{13.2}
$$

Here p_1 is a projection onto the first coordinate and σ is a covering mapping of degree $d = \deg f|_{\gamma_1} = \deg f|_{\gamma_2}$ defined by the formula $\sigma(z) = z^d$. Thus

$$
g \circ \phi_k(z,t) = z^d. \tag{13.3}
$$

We can also assume that $N_1 \cap N_2 = \varnothing$. To simplify notations for each pair $a, b \in [-2,2]$ we denote

$$
N_k^{[a,b]} = \phi_k(S^1 \times [a,b]).
$$

Let $\mu : [-2,2] \to [0,1]$ be a C^∞ function such that $\mu[-2,-1] = 0$ and $\mu[1,2] = 1$. Then the Dehn twist t_{γ_k} along γ_k can be defined so that $t = t_{\gamma_1} \circ t_{\gamma_2}^{-1}$ will have the form:

$$
t(z,t) = \begin{cases} x, & x \in M \setminus (N_1 \cup N_2) \\ (ze^{2\pi i \mu(s)}, s), & x = \phi_k(z,s) \in N_k, k = 1,2. \end{cases} \tag{13.4}
$$

Now a Σ-homotopy $G : M \times [0,1] \to S^1$ between g and $g \circ t$ can be defined by the formula:

$$
G(x,t) = \begin{cases} g(x)\, e^{2\pi i d t}, & x \in X \setminus (N_1^{[0,1]} \cup N_2^{[0,1]}), \\ g \circ \phi_k(z\, e^{2\pi i \mu(s)\cdot t}, s), & x = \phi_k(z,s) \in N_k, k = 1,2. \\ g(x), & x \in M \setminus (X \cup N_1^{[-1,0]} \cup N_2^{[-1,0]}). \end{cases}
$$

Remark 13.2. A geometrical meaning of this formula is that the mapping G "moves" d times the part X between the curves γ_1 and γ_2 "around S^1" leaving the complement $M \setminus X$ fixed.

Let us verify, that G is in fact a Σ-homotopy connecting g with $g \circ t$.

Proof. It is clear that $G_0 = g$. Moreover, it follows from (13.1) and (13.2) that ϕ_1 preserves orientation of $S^1 \times [-2,2]$ while ϕ_2 reverses it. Hence by (13.4) we get $G_1 = g \circ t_{\gamma_1} \circ t_{\gamma_2}^{-1}$.

Evidently, the continuity of G will imply its smoothness. To prove that G is continuous we should verify that the second formula coincides with the first one on $N_1^{[1,2]} \cup N_2^{[1,2]}$ and with the third one on $N_1^{[-2,-1]} \cup N_2^{[-2,-1]}$.

Let $x = \phi_k(z,s) \in N_k^{[1,2]}$ for $k = 1,2$, then $\mu(s) = 1$, whence, using (13.3), we get

$$
g \circ \phi_k(z\, e^{2\pi i \mu(s)\cdot t}, s) = z^d\, e^{2\pi i d t} = g(x)\, e^{2\pi i d t}.
$$

Let now $x = \phi_k(z,s) \in N_k^{[-2,-1]}$ for $k = 1,2$, then $\mu(s) = 0$, whence

$$
g \circ \phi_i(z\, e^{2\pi i \mu(s)\cdot t}, s) = g \circ \phi_i(z,s) = g(x).
$$

Notice that for every point $x \in M$ there exists a neighborhood on which G_t differs from g by a diffeomorphism of either S^1 or M. Hence G_t is Morse for all $t \in [0,1]$, i.e. G is a Σ-homotopy. □

14. Proof of Lemma 8.6

Suppose that $h \in \mathcal{PM}(M)$ is generated by $\{t_l : l \in \mathcal{C}\}$ and such that the mappings f and $f \circ h$ are homotopic. We have to prove that h is in fact generated by $\{t_l : l \in \mathcal{C} \setminus \beta_1\}$.

Recall that $H_1(M, \partial M)$ is a free module generated by homology classes of $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$. Moreover, the matrix of ω in this basis has the form (9.2). Since h_* preserves this ω we may suppose that $h_* \in Sp_{2g}(\mathbb{Z})$.

Notice that $h_*[\alpha_1] = [\alpha_1]$, since α_1 is a level-set of f, whence h_* belongs to the stabilizer $\mathrm{St}([\alpha_1])$ of α_1 in $Sp_{2g}(\mathbb{Z})$.

Let t_γ be a Dehn twist along simple closed curve γ. Then it acts on $H_1(M, \partial M)$ by the following formula:

$$(t_\gamma)_*(x) = \omega\left([\gamma], x\right) \cdot [\gamma] + x, \qquad \forall x \in H_1(M), \tag{14.1}$$

thus it is a transvection along $[\gamma]$, see Eq. (9.1).

Hence the products of transvections μ_{ij}, η_{ij}, ν_{ij} defined by Formula (9.2) can be realized by products of Dehn twists. It follows from Theorem 8.7 that all these diffeomorphisms except for $\eta_{1i} = \eta_{i1}$ and ν_{i1} are f-admissible.

On the other hand, by Proposition 9.3, h_* is generated by the linear isomorphisms t_{α_i}, t_{β_i}, μ_{ij}, η_{ij}, ν_{ij}, except for t_{β_1} $\eta_{1i} = \eta_{i1}$ and ν_{i1}, where $i \neq j = 1, \dots, g$.

Hence, there exists an f-admissible diffeomorphism c of M which induces the same isomorphism of H as h_*. Then $t = c^{-1} \circ h$ belongs to $\mathcal{T}(M)$. □

Appendix. Proof of Main Theorem. Case $P = S^1$

We extend here our proof of Main Theorem given in [M] to the case when M is arbitrary and $P = S^1$.

Let $f, g : M \to S^1$ be two Morse mappings of same critical type, c be their common regular value, $\alpha = f^{-1}(c)$, and $\gamma = g^{-1}(c)$. By Lemma 5.2 we can assume that homomorphism $f_* = g_* : H_1(M) \to H_1(S^1)$ is onto and by Lemma 3.2 that α and γ are connected, i.e. SCCs.

Let us cut M along α and denote the obtained surface by $\widetilde{M}$. Let also $p : \widetilde{M} \to M$ be the factor-mapping, $\widetilde{f} : \widetilde{M} \to [0,1]$ the corresponding Morse function induced by f, $B_0 = \widetilde{f}^{-1}(0)$, $B_1 = \widetilde{f}^{-1}(1)$, and $B = B_0 \cup B_1$ (we use the notations of Section 3).

Claim 14.1. *If $\alpha = \gamma$, then $f \overset{\Sigma}{\sim} g$.*

Proof. Since f and g are homotopic, we can assume (by small Σ-homotopy) that they coincide near α. Then g also yields a Morse function $\widetilde{g} : M \to [0,1]$ which coincides near B with $\widetilde{f}$ and $K(\widetilde{f}) = K(\widetilde{g})$. By the $\mathbb{R}$-case of Main Theorem $\widetilde{f} \overset{\Sigma}{\sim} \widetilde{g}$ with respect to a neighborhood of B. Then this

Σ-homotopy yields a Σ-homotopy between f and g with respect to a neighborhood of α. $\square$

Suppose that $\alpha \neq \gamma$. Since f and g are homotopic, it follows that the restriction $f|_\gamma$ is null-homotopic. Then by Lemma 11.1 we can additionally assume that $\alpha \cap \gamma = \varnothing$.

In this case $\widetilde{\gamma} = p^{-1}(\gamma)$ separates $\widetilde{M}$ between B_0 and B_1. Using the method of Claim 11.2 we can construct a Morse function $\widetilde{f}_1 : \widetilde{M} \to [0,1]$ which coincides with $\widetilde{f}$ near $B_0 \cup B_1$, has critical type of $\widetilde{f}$, and such that $f_1^{-1}(\frac{1}{2}) = \widetilde{\gamma}$. Then $\widetilde{f}_1$ yields a Morse mapping $f_1 : M \to S^1$ which coincides with $\widetilde{f}$ in a neighborhood of α and such that $f_1^{-1}(p(\frac{1}{2})) = \gamma$. Thus α and γ are level-sets of f_1. Then by Claim 14.1 we get $f \overset{\Sigma}{\sim} f_1 \overset{\Sigma}{\sim} g$. $\square$

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